

SECTION - 1

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If **ONLY** the correct option is chosen;

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

Negative Marks : -1 In all other cases.

1. Consider a triangle Δ whose two sides lie on the x -axis and the line $x + y + 1 = 0$. If the orthocentre of Δ is $(1, 1)$, then the equation of the circle passing through the vertices of the triangle Δ is
- (A) $x^2 + y^2 - 3x + y = 0$ (B) $x^2 + y^2 + x + 3y = 0$
 (C) $x^2 + y^2 + 2y - 1 = 0$ (D) $x^2 + y^2 + x + y = 0$

Answer (B)

Sol. As we know mirror image of orthocentre lie on circumcircle.

Image of $(1, 1)$ in x -axis is $(1, -1)$

Image of $(1, 1)$ in $x + y + 1 = 0$ is $(-2, -2)$.

$\therefore$ The required circle will be passing through both $(1, -1)$ and $(-2, -2)$.

$\therefore$ Only $x^2 + y^2 + x + 3y = 0$ satisfy both.

2. The area of the region

$$\left\{ (x, y) : 0 \leq x \leq \frac{9}{4}, \quad 0 \leq y \leq 1, \quad x \geq 3y, \quad x + y \geq 2 \right\}$$

is

(A) $\frac{11}{32}$

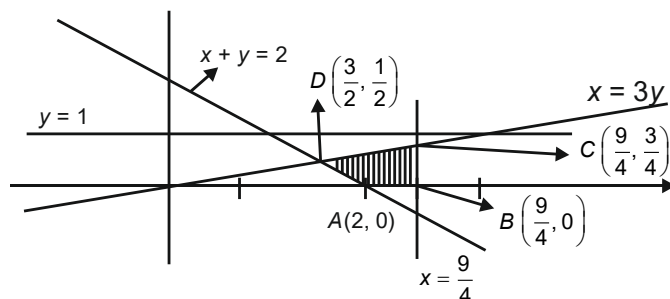
(B) $\frac{35}{96}$

(C) $\frac{37}{96}$

(D) $\frac{13}{32}$

Answer (A)

Sol. Rough sketch of required region is



∴ Required area is

Area of $\triangle ACD$ + Area of $\triangle ABC$

$$\text{i.e., } \frac{1}{4} + \frac{3}{32} = \frac{11}{32} \text{ sq. units}$$

3. Consider three sets $E_1 = \{1, 2, 3\}$, $F_1 = \{1, 3, 4\}$ and $G_1 = \{2, 3, 4, 5\}$. Two elements are chosen at random, without replacement, from the set E_1 , and let S_1 denote the set of these chosen elements. Let $E_2 = E_1 - S_1$ and $F_2 = F_1 \cup S_1$. Now two elements are chosen at random, without replacement, from the set F_2 and let S_2 denote the set of these chosen elements.

Let $G_2 = G_1 \cup S_2$. Finally, two elements are chosen at random, without replacement from the set G_2 and let S_3 denote the set of these chosen elements.

Let $E_3 = E_2 \cup S_3$. Given that $E_1 = E_3$, let p be the conditional probability of the event $S_1 = \{1, 2\}$. Then the value of p is

(A) $\frac{1}{5}$

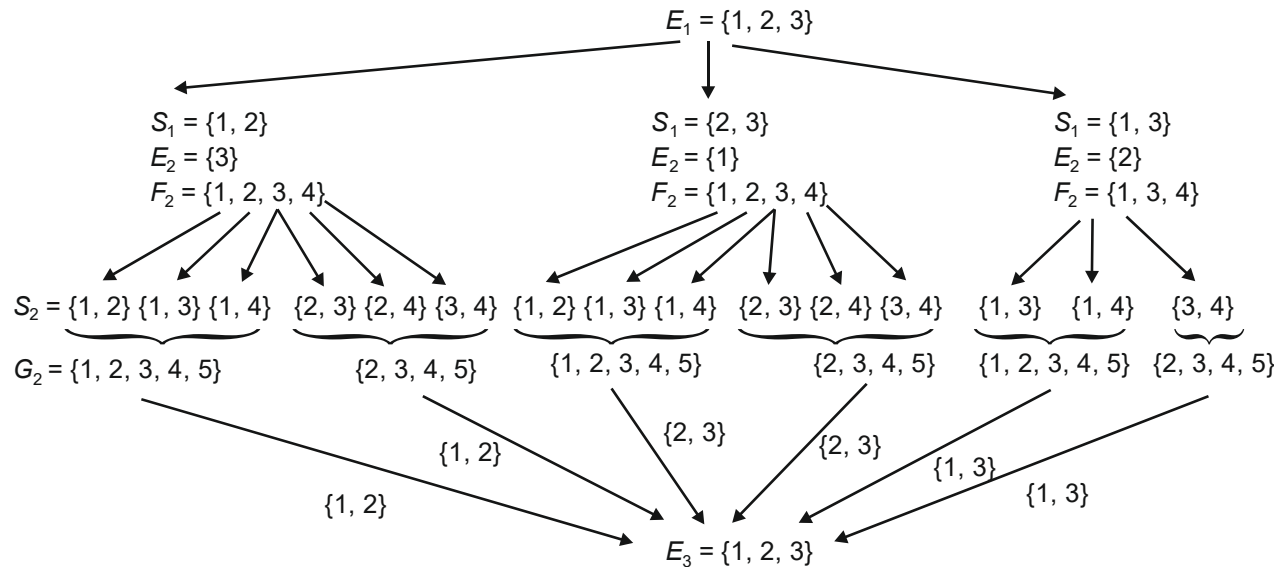
(B) $\frac{3}{5}$

(C) $\frac{1}{2}$

(D) $\frac{2}{5}$

Answer (A)

Sol. We will follow the tree diagram,



$$P(E_1 = E_3) = \frac{1}{3} \left[\frac{1}{2} \times \frac{1}{10} + \frac{1}{2} \times 0 + \frac{1}{2} \times \frac{1}{10} + \frac{1}{2} \times \frac{1}{6} + \frac{2}{3} \times \frac{1}{10} + \frac{1}{3} \times 0 \right]$$

$$= \frac{1}{3} \left[\frac{1}{4} \right]$$

$$\text{Required probability} = \frac{\frac{1}{3} \left[\frac{1}{2} \times \frac{1}{10} \right]}{\frac{1}{3} \times \frac{1}{4}} = \frac{1}{5}$$

4. Let $\theta_1, \theta_2, \dots, \theta_{10}$ be positive valued angles (in radian) such that $\theta_1 + \theta_2 + \dots + \theta_{10} = 2\pi$. Define the complex numbers $z_1 = e^{i\theta_1}, z_k = z_{k-1}e^{i\theta_k}$ for $k = 2, 3, \dots, 10$, where $i = \sqrt{-1}$. Consider the statement P and Q given below:

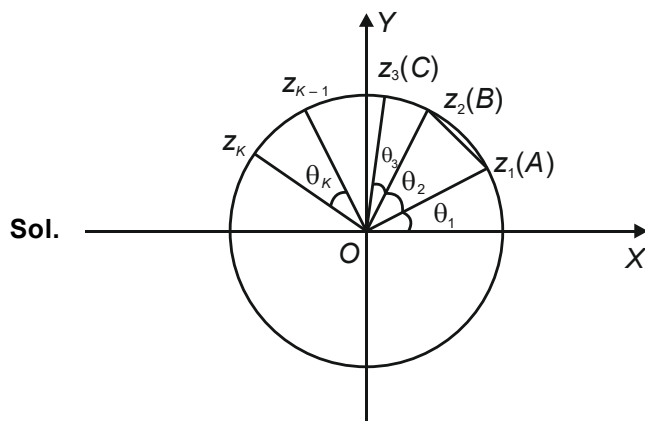
$$P: |z_2 - z_1| + |z_3 - z_2| + \dots + |z_{10} - z_9| + |z_1 - z_{10}| \leq 2\pi$$

$$Q: |z_2^2 - z_1^2| + |z_3^2 - z_2^2| + \dots + |z_{10}^2 - z_9^2| + |z_1^2 - z_{10}^2| \leq 4\pi$$

Then,

- (A) P is **TRUE** and Q is **FALSE**
 (B) Q is **TRUE** and P is **FALSE**
 (C) Both P and Q are **TRUE**
 (D) Both P and Q are **FALSE**

Answer (C)



$$|z_2 - z_1| = \text{length of line } AB \leq \text{length of arc } AB$$

$$|z_3 - z_2| = \text{length of line } BC \leq \text{length of arc } BC$$

$\therefore$ Sum of length of these 10 lines $\leq$ Sum of length of arcs (i.e. 2π)

(As $(\theta_1 + \theta_2 + \dots + \theta_{10}) = 2\pi$)

$$\therefore |z_2 - z_1| + |z_3 - z_2| + \dots + |z_1 - z_{10}| \leq 2\pi$$

$$\text{And } |z_k^2 - z_{k-1}^2| = |z_k - z_{k-1}| |z_k + z_{k-1}|$$

$$\text{As we know } |z_k + z_{k-1}| \leq |z_k| + |z_{k-1}| \leq 2$$

$$|z_2^2 - z_1^2| + |z_3^2 - z_2^2| + \dots + |z_1^2 - z_{10}^2| \leq 2(|z_2 - z_1| + |z_3 - z_2| + \dots + |z_1 - z_{10}|)$$

$$\leq 2(2\pi)$$

$\therefore$ Both (P) and (Q) are true.

SECTION - 2

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numerical keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +2 If ONLY the correct numerical value is entered at the designated place;

Zero Marks : 0 In all other cases.

Question Stem for Question Nos. 5 and 6

Question Stem

Three numbers are chosen at random, one after another with replacement, from the set $S = \{1, 2, 3, \dots, 100\}$. Let p_1 be the probability that the maximum of chosen numbers is at least 81 and p_2 be the probability that the minimum of chosen numbers is at most 40.

5. The value of $\frac{625}{4} p_1$ is _____.

Answer (76.25)

Sol. For p_1 , we need to remove the cases when all three numbers are less than or equal to 80.

$$\text{So, } p_1 = 1 - \left(\frac{80}{100}\right)^3 = \frac{61}{125}$$

$$\text{So, } \frac{625}{4} p_1 = \frac{625}{4} \times \frac{61}{125} = \frac{305}{4} = 76.25$$

6. The value of $\frac{125}{4} p_2$ is _____.

Answer (24.50)

Sol. For p_2 , we need to remove the cases when all three numbers are greater than 40.

$$\text{So, } p_2 = 1 - \left(\frac{60}{100}\right)^3 = \frac{98}{125}$$

$$\text{So, } \frac{125}{4} p_2 = \frac{125}{4} \times \frac{98}{125} = 24.50$$

Question Stem for Question Nos. 7 and 8

Question Stem

Let α , β and γ be real numbers such that the system of linear equations

$$x + 2y + 3z = \alpha$$

$$4x + 5y + 6z = \beta$$

$$7x + 8y + 9z = \gamma - 1$$

is consistent. Let $|M|$ represent the determinant of the matrix

$$M = \begin{bmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Let P be the plane containing all those (α, β, γ) for which the above system of linear equations is consistent, and D be the **square** of the distance of the point $(0, 1, 0)$ from the plane P .

7. The value of $|M|$ is _____.

Answer (1)

8. The value of D is _____.

Answer (1.50)

Sol. Solution for Q 7 and 8

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$$

Given system of equation will be consistent even if $\alpha = \beta = \gamma - 1 = 0$, i.e. equations will form homogeneous system.

So, $\alpha = 0$, $\beta = 0$, $\gamma = 1$

$$M = \begin{vmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = -1(-1) = +1$$

As given equations are consistent

$$x + 2y + 3z - \alpha = 0 \quad \dots P_1$$

$$4x + 5y + 6z - \beta = 0 \quad \dots P_2$$

$$7x + 8y + 9z - (\gamma - 1) = 0 \quad \dots P_3$$

For some scalar λ and μ

$$\mu P_1 + \lambda P_2 = P_3$$

$$\mu(x + 2y + 3z - \alpha) + \lambda(4x + 5y + 6z - \beta) = 7x + 8y + 9z - (\gamma - 1)$$

Comparing coefficients

$$\mu + 4\lambda = 7, \quad 2\mu + 5\lambda = 8, \quad 3\mu + 6\lambda = 9$$

$\lambda = 2$ and $\mu = -1$ satisfy all these conditions

comparing constant terms,

$$-\alpha\mu - \beta\lambda = -(\gamma - 1)$$

$$\alpha - 2\beta + \gamma = 1$$

So equation of plane is

$$x - 2y + z = 1$$

$$\text{Distance from } (0, 1, 0) = \left| \frac{-2-1}{\sqrt{6}} \right| = \frac{3}{\sqrt{6}}$$

$$D = \left(\frac{3}{\sqrt{6}} \right)^2 = \frac{3}{2} = 1.50$$

Question Stem for Question Nos. 9 and 10

Question Stem

Consider the lines L_1 and L_2 defined by

$$L_1 : x\sqrt{2} + y - 1 = 0 \text{ and } L_2 : x\sqrt{2} - y + 1 = 0$$

For a fixed constant λ , let C be the locus of a point P such that the product of the distance of P from L_1 and the distance of P from L_2 is λ^2 . The line $y = 2x + 1$ meets C at two points R and S , where the distance between R and S is $\sqrt{270}$.

Let the perpendicular bisector of RS meet C at two distinct points R' and S' . Let D be the square of the distance between R' and S' .

9. The value of λ^2 is _____.

Answer (9)

10. The value of D is _____.

Answer (77.14)

Sol. Solution for Q 9 and 10

$$C : \left| \frac{x\sqrt{2} + y - 1}{\sqrt{3}} \right| \left| \frac{x\sqrt{2} - y + 1}{\sqrt{3}} \right| = \lambda^2$$

$$\Rightarrow C : |2x^2 - (y-1)^2| = 3\lambda^2$$

C cuts $y - 1 = 2x$ at $R(x_1, y_1)$ and $S(x_2, y_2)$

$$\text{So, } |2x^2 - 4x^2| = 3\lambda^2 \Rightarrow x = \pm \sqrt{\frac{3}{2}} |\lambda|$$

$$\text{So, } |x_1 - x_2| = \sqrt{6} |\lambda| \text{ and } |y_1 - y_2| = 2|x_1 - x_2| = 2\sqrt{6} |\lambda|$$

$$\therefore RS^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \Rightarrow 270 = 30\lambda^2 \Rightarrow \lambda^2 = 9$$

$$\therefore \text{Slope of } RS = 2 \text{ and mid-point of } RS \text{ is } \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \equiv (0, 1)$$

$$\text{So, } R'S' \equiv y - 1 = -\frac{1}{2}x$$

$$\text{Solving } y - 1 = -\frac{1}{2}x \text{ with 'C' we get } x^2 = \frac{12}{7}\lambda^2$$

$$\Rightarrow |x_1 - x_2| = 2\sqrt{\frac{12}{7}} |\lambda| \text{ and } |y_1 - y_2| = \frac{1}{2}|x_1 - x_2| = \sqrt{\frac{12}{7}} |\lambda|$$

$$\text{Hence, } D = (R'S')^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = \frac{12}{7} \cdot 9 \times 5 \approx 77.14$$

SECTION - 3

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

<i>Full Marks</i>	: +4	If only (all) the correct option(s) is(are) chosen;
<i>Partial Marks</i>	: +3	If all the four options are correct but ONLY three options are chosen;
<i>Partial Marks</i>	: +2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
<i>Partial Marks</i>	: +1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
<i>Zero Marks</i>	: 0	If unanswere;
<i>Negative Marks</i>	: -2	In all other cases.
- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
 - choosing ONLY (A), (B) and (D) will get +4 marks;
 - choosing ONLY (A) and (B) will get +2 marks;
 - choosing ONLY (A) and (D) will get +2 marks;
 - choosing ONLY (B) and (D) will get +2 marks;
 - choosing ONLY (A) will get +1 mark;
 - choosing ONLY (B) will get +1 mark;
 - choosing ONLY (D) will get +1 mark;
 - choosing no option(s) (i.e., the question is unanswered) will get 0 marks and
 - choosing any other option(s) will get -2 marks.

11. For any 3×3 matrix M , let $|M|$ denote the determinant of M . Let

$$E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } F = \begin{bmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{bmatrix}$$

If Q is a nonsingular matrix of order 3×3 , then which of the following statements is(are) **TRUE**?

(A) $F = PEP$ and $P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B) $|EQ + PFQ^{-1}| = |EQ| + |PFQ^{-1}|$

(C) $|(EF)^3| > |EF|^2$

(D) Sum of the diagonal entries of $P^{-1}EP + F$ is equal to the sum of diagonal entries of $E + P^{-1}FP$

Answer (A, B, D)

Sol. $\because P$ is formed from I by exchanging second and third row or by exchanging second and third column.

So, PA is a matrix formed from A by changing second and third row.

Similarly AP is a matrix formed from A by changing second and third column.

Hence, $\text{Tr}(PAP) = \text{Tr}(A) \quad \dots(1)$

(A) Clearly, $P.P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

and $PE = \begin{bmatrix} 1 & 2 & 3 \\ 8 & 13 & 18 \\ 2 & 3 & 4 \end{bmatrix} \Rightarrow PEP = \begin{bmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{bmatrix} = F$

$\Rightarrow PEP = F \Rightarrow PFP = E \quad \dots(2)$

(B) $\because |E| = |F| = 0$

So, $|EQ + PFQ^{-1}| = |PFPQ + PFQ^{-1}| = |P| |F| |PQ + Q^{-1}| = 0$

Also, $|EQ| + |PFQ^{-1}| = 0$

(C) From (2); $PFP = E$ and $|P| = -1$

So, $|F| = |E|$

Also, $|E| = 0 = |F|$

So, $|EF|^3 = 0 = |EF|^2$

(D) $\because P^2 = I \Rightarrow P^{-1} = P$

So, $\text{Tr}(P^{-1}EP + F) = \text{Tr}(PEP + F) = \text{Tr}(2F)$

Also $\text{Tr}(E + P^{-1}FP) = \text{Tr}(E + PFP) = \text{Tr}(2E)$

Given that $\text{Tr}(E) = \text{Tr}(F)$

$\Rightarrow \text{Tr}(2E) = \text{Tr}(2F)$

12. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}$$

Then which of the following statements is (are) **TRUE**?

(A) f is decreasing in the interval $(-2, -1)$ (B) f is increasing in the interval $(1, 2)$

(C) f is onto

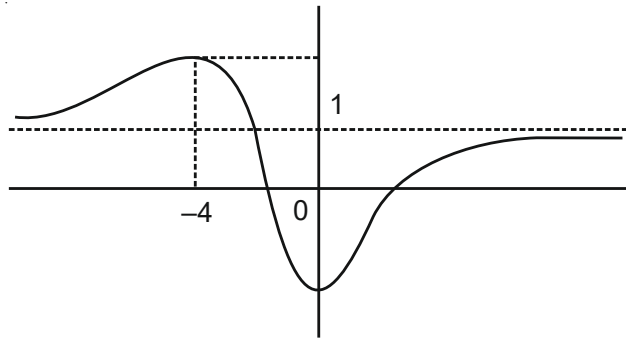
(D) Range of f is $\left[-\frac{3}{2}, 2\right]$

Answer (A, B)

Sol. $f(x) = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}$

$\Rightarrow f'(x) = \frac{5x(x+4)}{(x^2 + 2x + 4)^2}$

$\Rightarrow f(x)$ has local maxima at $x = -4$ and minima at $x = 0$



Range of $f(x)$ is $\left[-\frac{3}{2}, \frac{11}{6}\right]$

13. Let E, F and G be three events having probabilities

$$P(E) = \frac{1}{8}, P(F) = \frac{1}{6} \text{ and } P(G) = \frac{1}{4}, \text{ and } P(E \cap F \cap G) = \frac{1}{10}.$$

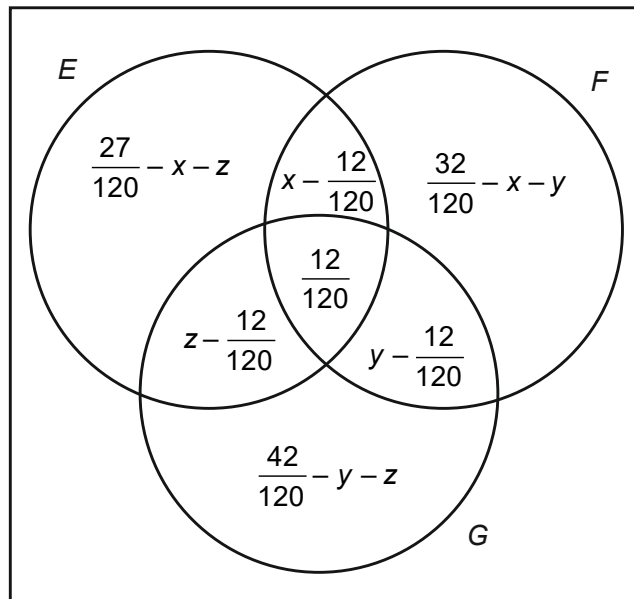
For any event H , if H^c denotes its complement, then which of the following statements is(are) **TRUE**?

- (A) $P(E \cap F \cap G^c) \leq \frac{1}{40}$ (B) $P(E^c \cap F \cap G) \leq \frac{1}{15}$
 (C) $P(E \cup F \cup G) \leq \frac{13}{24}$ (D) $P(E^c \cap F^c \cap G^c) \leq \frac{5}{12}$

Answer (A, B, C)

Sol. Let $P(E \cap F) = x, P(F \cap G) = y$ and $P(E \cap G) = z$

Clearly $x, y, z \geq \frac{1}{10}$



$$\therefore x + z \leq \frac{27}{120} \Rightarrow x, z \leq \frac{15}{120}$$

$$x + y \leq \frac{32}{120} \Rightarrow x, y \leq \frac{20}{120}$$

$$\text{and } y + z \leq \frac{42}{120} \Rightarrow y, z \leq \frac{30}{120}$$

$$\text{Now } P(E \cap F \cap G^c) = x - \frac{12}{120} \leq \frac{3}{120} = \frac{1}{40}$$

$$P(E^c \cap F \cap G) = y - \frac{12}{120} \leq \frac{80}{120} = \frac{1}{15}$$

$$P(E \cup F \cup G) \leq P(E) + P(F) + P(G) = \frac{13}{24}$$

$$\text{and } P(E^c \cap F^c \cap G^c) = 1 - P(E \cup F \cup G) \geq \frac{11}{24} \geq \frac{5}{12}$$

14. For any 3×3 matrix M , let $|M|$ denote the determinant of M . Let I be the 3×3 identity matrix. Let E and F be two 3×3 matrices such that $(I - EF)$ is invertible. If $G = (I - EF)^{-1}$, then which of the following statements is (are) **TRUE**?

- (A) $|FE| = |I - FE| |FGE|$ (B) $(I - FE)(I + FGE) = I$
 (C) $EFG = GEF$ (D) $(I - FE)(I - FGE) = I$

Answer (A, B, C)

Sol. $\therefore I - EF = G^{-1}$

$$\Rightarrow G - GEF = I \quad \dots(1)$$

$$\text{and } G - EFG = I \quad \dots(2)$$

Clearly $GEF = EFG$ (option C is correct)

$$\begin{aligned} \text{Also } (I - FE)(I + FGE) &= I - FE + FGE - FE + FGE \\ &= I - FE + FGE - F(G - I)E \\ &= I - FE + FGE - FGE + FE \\ &= I \text{ (option B is correct and D is incorrect)} \end{aligned}$$

$$\begin{aligned} \text{Now, } (I - FE)(I - FGE) &= I - FE - FGE + F(G - I)E \\ &= I - 2FE \end{aligned}$$

$$\Rightarrow (I - FE)(-FGE) = -FE$$

$$\Rightarrow |I - FE||FGE| = |FE|$$

15. For any positive integer n , let $S_n : (0, \infty) \rightarrow \mathbb{R}$ be defined by

$$S_n(x) = \sum_{k=1}^n \cot^{-1} \left(\frac{1 + k(k+1)x^2}{x} \right)$$

where for any $x \in \mathbb{R}$, $\cot^{-1}(x) \in (0, \pi)$ and $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then which of the following statements is (are) **TRUE**?

(A) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1} \left(\frac{1 + 11x^2}{10x} \right)$, for all $x > 0$

(B) $\lim_{n \rightarrow \infty} \cot(S_n(x)) = x$, for all $x > 0$

(C) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0, \infty)$

(D) $\tan(S_n(x)) \leq \frac{1}{2}$, for all $n \geq 1$ and $x > 0$

Answer (A, B)

$$\text{Sol. } S_n(x) = \sum_{k=1}^n \tan^{-1} \left(\frac{(k+1)x - kx}{1 + kx \cdot (k+1)x} \right)$$

$$= \sum_{k=1}^n \left(\tan^{-1}(k+1)x - \tan^{-1} kx \right)$$

$$= \tan^{-1}(n+1)x - \tan^{-1} x = \tan^{-1} \left(\frac{nx}{1 + (n+1)x^2} \right)$$

$$\text{Now (A) } S_{10}(x) = \tan^{-1} \left(\frac{10x}{1 + 11x^2} \right) = \frac{\pi}{2} - \tan^{-1} \left(\frac{1 + 11x^2}{10x} \right)$$

$$\text{(B) } \lim_{n \rightarrow \infty} \cot(S_n(x)) = \cot \left(\tan^{-1} \left(\frac{x}{x^2} \right) \right) = x$$

$$\text{(C) } S_3(x) = \frac{\pi}{4} \Rightarrow \frac{3x}{1 + 4x^2} = 1 \Rightarrow 4x^2 - 3x + 1 = 0 \text{ has no real root.}$$

$$\text{(D) For } x = 1, \tan(S_n(x)) = \frac{n}{n+2} \text{ which is greater than } \frac{1}{2} \text{ for } n \geq 3 \text{ so this option is incorrect.}$$

16. For any complex number $w = c + id$, let $\arg(w) \in (-\pi, \pi]$, where $i = \sqrt{-1}$. Let α and β be real numbers such

that for all complex numbers $z = x + iy$ satisfying $\arg\left(\frac{z + \alpha}{z + \beta}\right) = \frac{\pi}{4}$, the ordered pair (x, y) lies on the circle

$$x^2 + y^2 + 5x - 3y + 4 = 0$$

Then which of the following statements is (are) **TRUE**?

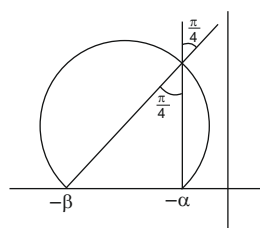
$$\text{(A) } \alpha = -1 \quad \text{(B) } \alpha\beta = 4$$

$$\text{(C) } \alpha\beta = -4 \quad \text{(D) } \beta = 4$$

Answer (B, D)

Sol. Circle $x^2 + y^2 + 5x - 3y + 4 = 0$ cuts the real axis (x-axis) at $(-4, 0)$, $(-1, 0)$

Clearly $\alpha = 1$ and $\beta = 4$



SECTION - 4

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If ONLY the correct integer is entered.

Zero Marks : 0 In all other cases.

17. For $x \in \mathbb{R}$, the number of real roots of the equation

$$3x^2 - 4|x^2 - 1| + x - 1 = 0$$

is _____.

Answer (4)

Sol. $3x^2 - 4|x^2 - 1| + x - 1 = 0$

Let $x \in [-1, 1]$

$$\Rightarrow 3x^2 - 4(-x^2 + 1) + x - 1 = 0$$

$$\Rightarrow 3x^2 + 4x^2 - 4 + x - 1 = 0$$

$$\Rightarrow 7x^2 + x - 5 = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1 + 140}}{2}$$

Both values acceptable

Let $x \in (-\infty, -1) \cup (1, \infty)$

$$x^2 - 4(x^2 - 1) + x - 1 = 0$$

$$\Rightarrow x^2 - x - 3 = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{1 + 12}}{2}$$

Again both are acceptable

Hence total number of solution = 4

18. In a triangle ABC , let $AB = \sqrt{23}$, and $BC = 3$ and $CA = 4$. Then the value of

$$\frac{\cot A + \cot C}{\cot B}$$

is _____.

Answer (2)

Sol. With standard notations

Given : $c = \sqrt{23}$, $a = 3$, $b = 4$

$$\text{Now } \frac{\cot A + \cot C}{\cot B} = \frac{\frac{\cos A}{\sin A} + \frac{\cos C}{\sin C}}{\frac{\cos B}{\sin B}}$$

$$= \frac{\frac{b^2 + c^2 - a^2}{2bc \sin A} + \frac{a^2 + b^2 - c^2}{2ab \sin C}}{\frac{c^2 + a^2 - b^2}{2ac \sin B}}$$

$$= \frac{\frac{b^2 + c^2 - a^2}{4\Delta} + \frac{a^2 + b^2 - c^2}{4\Delta}}{\frac{c^2 + a^2 - b^2}{4\Delta}} = \frac{2b^2}{a^2 + c^2 - b^2} = 2$$

19. Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in three-dimensional space, where $\vec{u}$ and $\vec{v}$ are unit vectors which are not perpendicular to each other and

$$\vec{u} \cdot \vec{w} = 1, \quad \vec{v} \cdot \vec{w} = 1, \quad \vec{w} \cdot \vec{w} = 4$$

If the volume of the parallelepiped, whose adjacent sides are represented by the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, is $\sqrt{2}$, then the value of $|3\vec{u} + 5\vec{v}|$ is _____.

Answer (7)

Sol. Given $[\vec{u} \ \vec{v} \ \vec{w}] = \sqrt{2}$

$$\text{Also } [\vec{u} \ \vec{v} \ \vec{w}]^2 = \begin{vmatrix} \vec{u} \cdot \vec{u} & \vec{u} \cdot \vec{v} & \vec{u} \cdot \vec{w} \\ \vec{v} \cdot \vec{u} & \vec{v} \cdot \vec{v} & \vec{v} \cdot \vec{w} \\ \vec{w} \cdot \vec{u} & \vec{w} \cdot \vec{v} & \vec{w} \cdot \vec{w} \end{vmatrix} = 2$$

Let $\vec{u} \cdot \vec{v} = k$ and substitute rest values, we get

$$\begin{vmatrix} 1 & K & 1 \\ K & 1 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 2$$

$$\Rightarrow 4K^2 - 2K = 0$$

$$\Rightarrow \vec{u} \cdot \vec{v} = 0 \quad \text{or} \quad \vec{u} \cdot \vec{v} = \frac{1}{2}$$

(rejected)

$$\therefore \vec{u} \cdot \vec{v} = \frac{1}{2}$$

$$|3\vec{u} + 5\vec{v}|^2 = 9 + 25 + 30 \times \frac{1}{2} = 49$$

$$\Rightarrow |3\vec{u} + 5\vec{v}| = 7$$