

PART-III : MATHEMATICS

SECTION - 1 (Maximum Marks : 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

<i>Full Marks</i>	: +4	If only (all) the correct option(s) is(are) chosen;
<i>Partial Marks</i>	: +3	If all the four options are correct but ONLY three options are chosen;
<i>Partial Marks</i>	: +2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
<i>Partial Marks</i>	: +1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
<i>Zero Marks</i>	: 0	If unanswered;
<i>Negative Marks</i>	: - 2	In all other cases.
- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
 - choosing ONLY (A), (B) and (D) will get +4 marks;
 - choosing ONLY (A) and (B) will get +2 marks;
 - choosing ONLY (A) and (D) will get +2 marks;
 - choosing ONLY (B) and (D) will get +2 marks;
 - choosing ONLY (A) will get +1 mark;
 - choosing ONLY (B) will get +1 mark;
 - choosing ONLY (D) will get +1 mark;
 - choosing no option(s) (i.e. the question is unanswered) will get 0 marks and
 - choosing any other option(s) will get "2 marks.

1. Let $S_1 = \{(i, j, k) : i, j, k \in \{1, 2, \dots, 10\}\},$

$$S_2 = \{(i, j) : 1 \leq i < j + 2 \leq 10, i, j \in \{1, 2, \dots, 10\}\},$$

$$S_3 = \{(i, j, k, l) : 1 \leq i < j < k < l, i, j, k, l \in \{1, 2, \dots, 10\}\},$$

and $S_4 = \{(i, j, k, l) : i, j, k \text{ and } l \text{ are distinct elements in } \{1, 2, \dots, 10\}\}.$

If the total number of elements in the set S_r is n_r , $r = 1, 2, 3, 4$, then which of the following statements is (are) TRUE?

(A) $n_1 = 1000$

(B) $n_2 = 44$

(C) $n_3 = 220$

(D) $\frac{n_4}{12} = 420$

Answer (A,B,D)

Sol. Number of elements in $S_1 = 10 \times 10 \times 10 = 1000$

$$\text{Number of elements in } S_2 = 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 = 44$$

$$\text{Number of elements in } S_3 = {}^{10}C_4 = 210$$

$$\text{Number of elements in } S_4 = {}^{10}P_4 = 210 \times 4! = 5040$$

2. Consider a triangle PQR having sides of lengths p, q and r opposite to the angles P, Q and R , respectively. Then which of the following statements is (are) TRUE ?

(A) $\cos P \geq 1 - \frac{p^2}{2qr}$

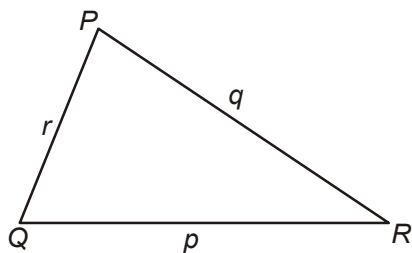
(B) $\cos R \geq \left(\frac{q-r}{p+q}\right)\cos P + \left(\frac{p-r}{p+q}\right)\cos Q$

(C) $\frac{q+r}{p} < 2\frac{\sqrt{\sin Q \sin R}}{\sin P}$

(D) If $p < q$ and $p < r$, then $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$

Answer (A,B)

Sol.



$$\cos P = \frac{q^2 + r^2 - p^2}{2qr} \quad \text{and} \quad \frac{q^2 + r^2}{2} \geq \sqrt{q^2 \cdot r^2} \quad (\text{AM} \geq \text{GM})$$

$$\Rightarrow q^2 + r^2 \geq 2qr$$

$$\text{So, } \cos P \geq \frac{2qr - p^2}{2qr}$$

$$\cos P \geq 1 - \frac{p^2}{2qr} \quad (\text{A})$$

$$\begin{aligned} \text{(B)} \quad \frac{(q-r)\cos P + (p-r)\cos Q}{p+q} &= \frac{(q\cos P + p\cos Q) - r(\cos P + \cos Q)}{p+q} \\ &= \frac{r(1 - \cos P - \cos Q)}{p+q} = \frac{r(q - p\cos R) - (p - q\cos R)}{p+q} = \frac{(r-p-q) + (p+q)\cos R}{p+q} \end{aligned}$$

$$= \cos R + \frac{r-q-p}{p+q} \leq \cos R \quad (\because r < p+q)$$

$$\text{(C)} \quad \frac{q+r}{p} = \frac{\sin Q + \sin R}{\sin P} \geq \frac{2\sqrt{\sin Q \cdot \sin R}}{\sin P}$$

$$\text{(D)} \quad \text{If } p < q \text{ and } q < r$$

So, p is the smallest side, therefore one of Q or R can be obtuse

So, one of $\cos Q$ or $\cos R$ can be negative

Therefore $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$ cannot hold always.

3. Let $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \mathbb{R}$ be a continuous function such that $f(0) = 1$ and $\int_0^{\frac{\pi}{3}} f(t) dt = 0$

Then which of the following statements is (are) TRUE?

- (A) The equation $f(x) - 3 \cos 3x = 0$ has at least one solution in $\left(0, \frac{\pi}{3}\right)$
- (B) The equation $f(x) - 3 \sin 3x = -\frac{6}{\pi}$ has at least one solution in $\left(0, \frac{\pi}{3}\right)$

(C) $\lim_{x \rightarrow 0} \frac{x \int_0^x f(t) dt}{1 - e^{x^2}} = -1$

(D) $\lim_{x \rightarrow 0} \frac{\sin x \int_0^x f(t) dt}{x^2} = -1$

Answer (A,B,C)

Sol. $f(0) = 1$, $\int_0^{\frac{\pi}{3}} f(t) dt = 0$

(A) Consider a function $g(x) = \int_0^x f(t) dt - \sin 3x$

$g(x)$ is continuous and differentiable function

and $g(0) = 0$

$$g\left(\frac{\pi}{3}\right) = 0$$

By Rolle's theorem $g'(x) = 0$ has at least one solution in $\left(0, \frac{\pi}{3}\right)$

$$f(x) - 3 \cos 3x = 0 \text{ for some } x \in \left(0, \frac{\pi}{3}\right)$$

(B) Consider a function

$$h(x) = \int_0^x f(t) dt + \cos 3x + \frac{6}{\pi} x$$

$h(x)$ is continuous and differentiable function

and $h(0) = 1$

$$h\left(\frac{\pi}{3}\right) = 1$$

By Rolle's theorem $h'(x) = 0$ for at least one $x \in \left(0, \frac{\pi}{3}\right)$

$$f(x) - 3 \sin 3x + \frac{6}{\pi} = 0 \text{ for some } x \in \left(0, \frac{\pi}{3}\right)$$

(C) $\lim_{x \rightarrow 0} \frac{x \int_0^x f(t) dt}{1 - e^{x^2}}, \left(\frac{0}{0} \text{ form}\right)$

By L' Hopital rule

$$\lim_{x \rightarrow 0} \frac{xf(x) + \int_0^x f(t)dt}{-2xe^{x^2}}, \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \rightarrow 0} \frac{xf'(x) + f(x) + f(x)}{-4x^2e^{x^2} - 2e^{x^2}} = \frac{0 + 2f(0)}{-0 - 2} = -1$$

$$(D) \lim_{x \rightarrow 0} \frac{\sin x \int_0^x f(t)dt}{x^2}, \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \rightarrow 0} \frac{\sin x \cdot f(x) + \cos x \int_0^x f(t)dt}{2x}$$

$$= \lim_{x \rightarrow 0} \frac{\left(\cos x \cdot f(x) + \sin x \cdot f'(x) + \cos x \cdot f(x) - \sin x \cdot \int_0^x f(t)dt\right)}{2}$$

$$= \frac{1 + 0 + 1 - 0}{2}$$

$$= 1$$

4. For any real numbers α and β , let $y_{\alpha\beta}(x)$, $x \in \mathbb{R}$, be the solution of the differential equation $\frac{dy}{dx} + \alpha y = xe^{\beta x}$, $y(1) = 1$

Let $S = \{y_{\alpha\beta}(x) : \alpha, \beta \in \mathbb{R}\}$. Then which of the following functions belong(s) to the set S ?

- (A) $f(x) = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$
- (B) $f(x) = -\frac{x^2}{2}e^{-x} + \left(e + \frac{1}{2}\right)e^{-x}$
- (C) $f(x) = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + \left(e - \frac{e^2}{4}\right)e^{-x}$
- (D) $f(x) = \frac{e^x}{2}\left(\frac{1}{2} - x\right) + \left(e + \frac{e^2}{4}\right)e^{-x}$

Answer (A, C)

Sol. $\frac{dy}{dx} + \alpha y = xe^{\beta x}$

Integrating factor (I.F.) = $e^{\int \alpha dx} = e^{\alpha x}$

So, the solution is $y \cdot e^{\alpha x} = \int xe^{\beta x} \cdot e^{\alpha x} dx$

$$ye^{\alpha x} = \int xe^{(\alpha+\beta)x} dx$$

If $\alpha + \beta \neq 0$

$$ye^{\alpha x} = x \frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)} - \frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)^2} + C$$

$$y = \frac{xe^{\beta x}}{(\alpha+\beta)} - \frac{e^{\beta x}}{(\alpha+\beta)^2} + Ce^{-\alpha x}$$

$$y = \frac{e^{\beta x}}{(\alpha+\beta)} \left(x - \frac{1}{\alpha+\beta} \right) + Ce^{-\alpha x} \quad \dots (i)$$

Put $\alpha = \beta = 1$ in (i)

$$y = \frac{e^x}{2} \left(x - \frac{1}{2} \right) + Ce^{-x}$$

$$y(1) = 1$$

$$1 = \frac{e}{2} \times \frac{1}{2} + \frac{C}{e} \Rightarrow C = e - \frac{e^2}{4}$$

$$\text{So, } y = \frac{e^x}{2} \left(x - \frac{1}{2} \right) + \left(e - \frac{e^2}{4} \right) e^{-x}$$

If $\alpha + \beta = 0$ & $\alpha = 1$

$$\frac{dy}{dx} + y = xe^{-x}$$

$$\text{I.F.} = e^x$$

$$ye^x = \int x dx$$

$$ye^x = \frac{x^2}{2} + C$$

$$y = \frac{x^2}{2} e^{-x} + Ce^{-x}$$

$$y(1) = 1$$

$$1 = \frac{1}{2e} + \frac{C}{e} \Rightarrow C = e - \frac{1}{2}$$

$$y = \frac{x^2}{2} e^{-x} + \left(e - \frac{1}{2} \right) e^{-x}$$

5. Let O be the origin and $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\overrightarrow{OC} = \frac{1}{2}(\overrightarrow{OB} - \lambda\overrightarrow{OA})$ for some $\lambda > 0$. If

$$|\overrightarrow{OB} \times \overrightarrow{OC}| = \frac{9}{2}, \text{ then which of the following statements is (are) TRUE ?}$$

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is $-\frac{3}{2}$

(B) Area of the triangle OAB is $\frac{9}{2}$

(C) Area of the triangle ABC is $\frac{9}{2}$

(D) The acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\frac{\pi}{3}$

Answer (A,B,C)

Sol. $\vec{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$

$$\vec{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$$

$$\vec{OC} = \frac{1}{2}(\vec{OB} - \lambda\vec{OA})$$

$$\vec{OB} \times \vec{OC} = \vec{OB} \times \frac{1}{2}(\vec{OB} - \lambda\vec{OA}) = -\frac{\lambda}{2}\vec{OB} \times \vec{OA} = \frac{\lambda}{2}(\vec{OA} \times \vec{OB})$$

$$\text{Now, } \vec{OA} \times \vec{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{vmatrix} = 6\hat{i} - 3\hat{j} - 6\hat{k}$$

$$\text{So, } \vec{OB} \times \vec{OC} = \frac{3\lambda}{2}(2\hat{i} - \hat{j} - 2\hat{k})$$

$$|\vec{OB} \times \vec{OC}| = \left| \frac{9\lambda}{2} \right| = \frac{9}{2}$$

$$\text{So, } \lambda = 1 \quad (\because \lambda > 0)$$

$$\vec{OC} = \frac{1}{2}(\vec{OB} - \vec{OA})$$

$$\vec{OC} = \frac{1}{2}(-\hat{i} - 4\hat{j} + \hat{k})$$

$$(A) \text{ Projection of } \vec{OC} \text{ on } \vec{OA} = \frac{\vec{OC} \cdot \vec{OA}}{|\vec{OA}|} = \frac{\frac{1}{2}(-2-8+1)}{3} = -\frac{3}{2}$$

$$(B) \text{ Area of the triangle } OAB = \frac{1}{2}|\vec{OA} \times \vec{OB}| = \frac{9}{2}$$

$$(C) \text{ Area of the triangle } ABC \text{ is } = \frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2} \left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -4 & 1 \\ -\frac{5}{2} & -4 & -\frac{1}{2} \end{vmatrix} \right| = \frac{1}{2}|6\hat{i} - 3\hat{j} - 6\hat{k}| = \frac{9}{2}$$

$$(D) \text{ Acute angle between the diagonals of the parallelogram with adjacent sides } \vec{OA} \text{ and } \vec{OC} = \theta$$

$$\frac{(\vec{OA} + \vec{OC}) \cdot (\vec{OA} - \vec{OC})}{|\vec{OA} + \vec{OC}| |\vec{OA} - \vec{OC}|} = \cos \theta$$

$$\cos \theta = \frac{\left(\frac{3}{2}\hat{i} + \frac{3}{2}\hat{k}\right) \cdot \left(\frac{5}{2}\hat{i} + 4\hat{j} + \frac{1}{2}\hat{k}\right)}{\frac{3}{2}\sqrt{2} \times \sqrt{\frac{90}{4}}} = \frac{18}{3\sqrt{2}\sqrt{90}}$$

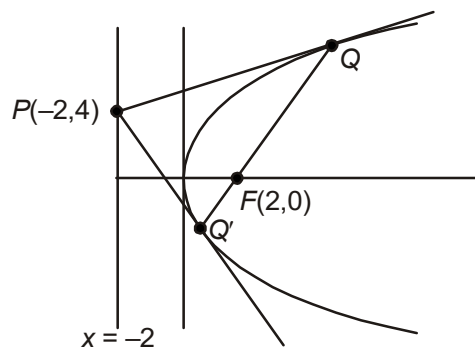
$$\theta \neq \frac{\pi}{3}$$

6. Let E denote the parabola $y^2 = 8x$. Let $P = (-2, 4)$, and let Q and Q' be two distinct points on E such that the lines PQ and PQ' are tangents to E . Let F be the focus of E . Then which of the following statements is (are) TRUE ?
- (A) The triangle PFQ is a right-angled triangle
- (B) The triangle QPQ' is a right-angled triangle
- (C) The distance between P and F is $5\sqrt{2}$
- (D) F lies on the line joining Q and Q'

Answer (A,B,D)

Sol. $E : y^2 = 8x$

$P : (-2, 4)$



Point $P(-2, 4)$ lies on directrix ($x = -2$) of parabola $y^2 = 8x$

So, $\angle QPQ' = \frac{\pi}{2}$ and chord QQ' is a focal chord and segment PQ subtends right angle at the focus.

$$\text{Slope of } QQ' = \frac{2}{t_1 + t_2} = 1$$

$$\text{Slope of } PF = -1$$

$$PF = 4\sqrt{2}$$

SECTION - 2 (Maximum Marks : 12)

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +2 If ONLY the correct numerical value is entered.

Zero Marks : 0 In all other cases.

Question Stem for Question Nos. 7 and 8

Consider the region $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0 \text{ and } y^2 \leq 4 - x\}$. Let F be the family of all circles that are contained in R and have centers on the x -axis. Let C be the circle that has largest radius among the circles in F . Let (α, β) be a point where the circle C meets the curve $y^2 = 4 - x$.

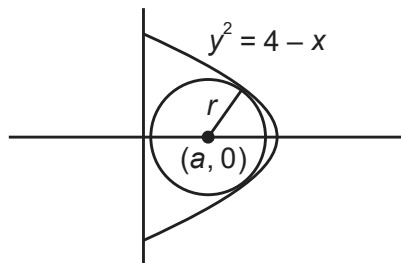
7. The radius of the circle C is _____.

Answer (1.50)

8. The value of α is _____.

Answer (2.00)

Sol. For comprehension Q7 & Q8



Let the circle be,

$$(x - a)^2 + y^2 = r^2$$

Solving it with parabola

$$y^2 = 4 - x \text{ we get}$$

$$(x - a)^2 + 4 - x = r^2$$

$$\Rightarrow x^2 - x(2a + 1) + (a^2 + 4 - r^2) = 0 \quad \dots(1)$$

$$D = 0$$

$$\Rightarrow 4r^2 + 4a - 15 = 0$$

Clearly $a \geq r$

$$\text{So } 4r^2 + 4r - 15 \leq 0$$

$$\Rightarrow r_{\max} = \frac{3}{2} = a$$

Radius of circle C is $\frac{3}{2}$

$$\text{From (1) } x^2 - 4x + 4 = 0$$

$$\Rightarrow x = 2 = \alpha$$

Question Stem for Question Nos. 9 and 10

Let $f_1 : (0, \infty) \rightarrow \mathbb{R}$ and $f_2 : (0, \infty) \rightarrow \mathbb{R}$ be defined by $f_1(x) = \int_0^x \prod_{j=1}^{21} (t - j)^j dt$, $x > 0$

and $f_2(x) = 98(x - 1)^{50} - 600(x - 1)^{49} + 2450$, $x > 0$, where, for any positive integer n and real number $a_1, a_2, \dots, a_n$,

$\prod_{i=1}^n a_i$ denotes the product of $a_1, a_2, \dots, a_n$. Let m_i and n_i , respectively, denote the number of points of local minima and the number of points of local maxima of function f_i , $i = 1, 2$, in the interval $(0, \infty)$.

Solution for Q9 and 10

$$f_1'(x) = \prod_{j=1}^{21} (x-j)^j$$

$$f_1'(x) = (x-1)(x-2)^2(x-3)^3, \dots, (x-20)^{20}(x-21)^{21}$$

Checking the sign scheme of $f_1'(x)$ at $x = 1, 2, 3, \dots, 21$, we get

$f_1(x)$ has local minima at $x = 1, 5, 9, 13, 17, 21$ and local maxima at $x = 3, 7, 11, 15, 19$

$$\Rightarrow m_1 = 6, n_1 = 5$$

$$f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450$$

$$f_2'(x) = 98 \times 50(x-1)^{49} - 600 \times 49 \times (x-1)^{48}$$

$$= 98 \times 50 \times (x-1)^{48} (x-7)$$

$f_2(x)$ has local minimum at $x = 7$ and no local maximum.

$$\Rightarrow m_2 = 1, n_2 = 0$$

9. The value of $2m_1 + 3n_1 + m_1n_1$ is _____.

Answer (57.00)

Sol. $2m_1 + 3n_1 + m_1n_1$

$$= 2 \times 6 + 3 \times 5 + 6 \times 5$$

$$= 57$$

10. The value of $6m_2 + 4n_2 + 8m_2n_2$ is _____.

Answer (06.00)

Sol. $6m_2 + 4n_2 + 8m_2n_2$

$$= 6 \times 1 + 4 \times 0 + 8 \times 1 \times 0 = 6$$

Question Stem for Question Nos. 11 and 12

Let $g_i : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \rightarrow \mathbb{R}, i = 1, 2$, and $f : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \rightarrow \mathbb{R}$ be functions such that $g_1(x) = 1$, $g_2(x) = |4x - \pi|$ and $f(x) =$

$\sin^2 x$, for all $x \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right]$. Define $S_i = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x) \cdot g_i(x) dx, i = 1, 2$

11. The value of $\frac{16S_1}{\pi}$ is _____.

Answer (02.00)

Sol. $S_1 = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \sin^2 x \cdot 1 dx$

$$= \frac{1}{2} \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} (1 - \cos 2x) dx$$

$$= \frac{1}{2} \left(x - \frac{\sin 2x}{2} \right) \Big|_{\frac{\pi}{8}}^{\frac{3\pi}{8}}$$

$$S_1 = \frac{1}{2} \left(\frac{\pi}{4} - 0 \right) = \frac{\pi}{8}$$

$$\Rightarrow \frac{16S_1}{\pi} = 2$$

12. The value of $\frac{48S_2}{\pi^2}$ is _____.

Answer (01.50)

Sol. $S_2 = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \sin^2 x \cdot |4x - \pi| dx$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} 4 \sin^2 x \left| x - \frac{\pi}{4} \right| dx$$

Let $x - \frac{\pi}{4} = t \Rightarrow dx = dt$

$$S_2 = \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} 4 \sin^2 \left(\frac{\pi}{4} + t \right) |t| dt$$

$$= \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} 2(1 - \cos 2\left(\frac{\pi}{4} + t\right)) |t| dt$$

$$= \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} (2 + 2 \sin 2t) |t| dt$$

$$= 2 \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} |t| dt + 2 \int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} |t| \sin(2t) dt$$

$$= 4 \int_0^{\frac{\pi}{8}} t dt + 0$$

$$S_2 = 2t^2 \Big|_0^{\frac{\pi}{8}} = \frac{\pi^2}{32}$$

$$\Rightarrow \frac{48S_2}{\pi^2} = \frac{3}{2}$$

SECTION - 3 (Maximum Marks : 12)

- This section contains **TWO (02) paragraphs**. Based on each paragraph, there are **TWO (02) questions**.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If ONLY the correct option is chosen;

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

Negative Marks : -1 In all other cases.

Paragraph

Let $M = \{(x, y) \in R \times R : x^2 + y^2 \leq r^2\}$, where $r > 0$. Consider the geometric progression $a_n = \frac{1}{2^{n-1}}$, $n = 1, 2, 3, \dots$. Let $S_0 = 0$ and, for $n \geq 1$, let S_n denote the sum of the first n terms of this progression. For $n \geq 1$, let C_n denote the circle with center $(S_{n-1}, 0)$ and radius a_n , and D_n denote the circle with center (S_{n-1}, S_{n-1}) and radius a_n .

13. Consider M with $r = \frac{1025}{513}$. Let k be the number of all those circles C_n that are inside M . Let l be the maximum possible number of circles among these k circles such that no two circles intersect. Then
- (A) $k + 2l = 22$
 (B) $2k + l = 26$
 (C) $2k + 3l = 34$
 (D) $3k + 2l = 40$

Answer (D)

Sol. $\therefore a_n = \frac{1}{2^{n-1}}$ and $S_n = 2 \left(1 - \frac{1}{2^n} \right)$

For circles C_n to be inside M .

$$S_{n-1} + a_n < \frac{1025}{513}$$

$$\Rightarrow S_n < \frac{1025}{513}$$

$$\Rightarrow 1 - \frac{1}{2^n} < \frac{1025}{1026} = 1 - \frac{1}{1026}$$

$$\Rightarrow 2^n < 1026$$

$$\Rightarrow n \leq 10$$

$$\therefore \text{Number of circles inside be } 10 = K$$

Clearly alternate circles do not intersect each other i.e., C_1, C_3, C_5, C_7, C_9 do not intersect each other as well as C_2, C_4, C_6, C_8 and C_{10} do not intersect each other hence maximum 5 set of circles do not intersect each other.

$$\therefore l = 5$$

$$\therefore 3K + 2l = 40$$

$\therefore$ Option (D) is correct

14. Consider M with $r = \frac{(2^{199} - 1)\sqrt{2}}{2^{198}}$. The number of all those circles D_n that are inside M is
- (A) 198 (B) 199
(C) 200 (D) 201

Answer (B)

Sol. $\therefore r = \frac{(2^{199} - 1)\sqrt{2}}{2^{198}}$

Now, $\sqrt{2} S_{n-1} + a_n < \left(\frac{2^{199} - 1}{2^{198}} \right) \sqrt{2}$

$$2\sqrt{2} \left(1 - \frac{1}{2^{n-1}} \right) + \frac{1}{2^{n-1}} < \left(\frac{2^{199} - 1}{2^{198}} \right) \sqrt{2}$$

$$\therefore 2\sqrt{2} - \frac{\sqrt{2}}{2^{n-2}} + \frac{1}{2^{n-1}} < 2\sqrt{2} - \frac{\sqrt{2}}{2^{198}}$$

$$\frac{1}{2^{n-2}} \left(\frac{1}{2} - \sqrt{2} \right) < -\frac{\sqrt{2}}{2^{198}}$$

$$\frac{2\sqrt{2} - 1}{2 \cdot 2^{n-2}} > \frac{\sqrt{2}}{2^{198}}$$

$$2^{n-2} < \left(2 - \frac{1}{\sqrt{2}} \right) 2^{197}$$

$$\therefore n \leq 199$$

$$\therefore \text{Number of circles} = 199$$

Option (B) is correct.

Paragraph

Let $\psi_1 : [0, \infty) \rightarrow \mathbb{R}$, $\psi_2 : [0, \infty) \rightarrow \mathbb{R}$, $f : [0, \infty) \rightarrow \mathbb{R}$ and $g : [0, \infty) \rightarrow \mathbb{R}$ be functions such that $f(0) = g(0) = 0$,

$$\psi_1(x) = e^{-x} + x, x \geq 0,$$

$$\psi_2(x) = x^2 - 2x - 2e^{-x} + 2, x \geq 0,$$

$$f(x) = \int_{-x}^x (|t| - t^2) e^{-t^2} dt, x > 0$$

and $g(x) = \int_0^{x^2} \sqrt{t} e^{-t} dt, x > 0.$

15. Which of the following statements is TRUE?

(A) $f(\sqrt{\ln 3}) + g(\sqrt{\ln 3}) = \frac{1}{3}$

(B) For every $x > 1$, there exists an $\alpha \in (1, x)$ such that $\Psi_1(x) = 1 + \alpha x$

(C) For every $x > 0$, there exists a $\beta \in (0, x)$ such that $\Psi_2(x) = 2x (\Psi_1(\beta) - 1)$

(D) f is an increasing function on the interval $\left[0, \frac{3}{2} \right]$

Answer (C)

Sol. $\therefore g(x) = \int_0^{x^2} \sqrt{t} e^{-t} dt, x > 0$

Let $t = u^2 \Rightarrow dt = 2u du$

$$\begin{aligned} \therefore g(x) &= \int_0^x u e^{-u^2} \cdot 2u du \\ &= 2 \int_0^x t^2 e^{-t^2} dt \end{aligned} \quad \dots(i)$$

and $f(x) = \int_{-x}^x (|t| - t^2) e^{-t^2} dt, x > 0$

$$\therefore f(x) = 2 \int_0^x (t - t^2) e^{-t^2} dt \quad \dots(ii)$$

From equation (i) + (ii) : $f(x) + g(x) = \int_0^x 2te^{-t^2} dt$

Let $t^2 = P \Rightarrow 2t dt = dP$

$$\begin{aligned} \therefore f(x) + g(x) &= \int_0^{x^2} e^{-P} dP = [-e^{-P}]_0^{x^2} \\ \therefore f(x) + g(x) &= 1 - e^{-x^2} \end{aligned} \quad \dots(iii)$$

$$\therefore f(\sqrt{\ln 3}) + g(\sqrt{\ln 3}) = 1 - e^{-\ln 3} = 1 - \frac{1}{3} = \frac{2}{3}$$

$\therefore$ Option (A) is incorrect.

From equation (ii) : $f'(x) = 2(x - x^2)e^{-x^2} = 2x(1 - x)e^{-x^2}$

$\therefore f(x)$ is increasing in $(0, 1)$

$\therefore$ Option (D) is incorrect

$$\therefore \Psi_1(x) = e^{-x} + x$$

$$\Rightarrow \Psi'_1(x) = 1 - e^{-x} < 1 \text{ for } x > 1$$

Then for $\alpha \in (1, x)$, $\Psi_1(x) = 1 + \alpha x$ does not true for $\alpha > 1$.

$\therefore$ Option (B) is incorrect

Now $\Psi_2(x) = x^2 - 2x - 2e^{-x} + 2$

$$\Rightarrow \Psi'_2(x) = 2x - 2 + 2e^{-x}$$

$$\therefore \Psi'_2(x) = 2\Psi_1(x) - 2$$

From LMVT

$$\frac{\Psi_2(x) - \Psi_2(0)}{x - 0} = \Psi'_2(\beta) \text{ for } \beta \in (0, x)$$

$$\Rightarrow \Psi_2(x) = 2x(\Psi_1(\beta) - 1)$$

$\therefore$ Option (C) is correct.

16. Which of the following statements is TRUE?

(A) $\Psi_1(x) \leq 1$, for all $x > 0$

(B) $\Psi_2(x) \leq 0$, for all $x > 0$

(C) $f(x) \geq 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5$, for all $x \in \left(0, \frac{1}{2}\right)$

(D) $g(x) \leq \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$, for all $x \in \left(0, \frac{1}{2}\right)$

Answer (D)

Sol. $\because \Psi_1(x) = e^{-x} + x$

and for all $x > 0$, $\Psi_1(x) > 1$

$\therefore$ (A) is not correct

$\Psi_1(x) = x^2 + 2 - 2(e^{-x} + x) > 0$ for $x > 0$

$\therefore$ (B) is not correct

Now, $\sqrt{t} e^{-t} = \sqrt{t} \left(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \infty\right)$

and $\sqrt{t} e^{-t} \leq t^{\frac{1}{2}} - t^{\frac{3}{2}} + \frac{1}{2}t^{\frac{5}{2}}$

$\therefore \int_0^{x^2} \sqrt{t} e^{-t} dt \leq \int_0^{x^2} \left(t^{\frac{1}{2}} - t^{\frac{3}{2}} + \frac{1}{2}t^{\frac{5}{2}}\right) dt$

$= \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$

$\therefore$ Option (D) is correct

and $f(x) = \int_{-x}^x (|t| - t^2)e^{-t^2} dt$

$= 2 \int_0^x (t - t^2)e^{-t^2} dt$

$= \int_0^x 2te^{-t^2} dt - 2 \int_0^x t^2 e^{-t^2} dt$

$= 1 - e^{-x^2} - 2 \int_0^x t^2 e^{-t^2} dt$

$\therefore f(x) \leq 1 - e^{-x^2} - 2 \int_0^x t^2 (1 - t^2) dt$

$= 1 - e^{-x^2} - 2 \frac{x^3}{3} + \frac{2}{5}x^5$ for all $x \in \left(0, \frac{1}{2}\right)$

$\therefore$ Option (C) is incorrect.

SECTION - 4 (Maximum Marks : 12)

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If ONLY the correct integer is entered;

Zero Marks : 0 In all other cases.

17. A number is chosen at random from the set $\{1, 2, 3, \dots, 2000\}$. Let p be the probability that the number is a multiple of 3 or a multiple of 7. Then the value of $500p$ is _____.

Answer (214)

Sol. $E =$ a number which is multiple of 3 or multiple of 7

$$n(E) = (3, 6, 9, \dots, 1998) + (7, 14, 21, \dots, 1995) - (21, 42, 63, \dots, 1995)$$

$$n(E) = 666 + 285 - 95$$

$$n(E) = 856$$

$$n(E) = 2000$$

$$P(E) = \frac{856}{2000}$$

$$P(E) \times 500 = \frac{856}{4} = 214$$

18. Let E be the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. For any three distinct points P , Q and Q' on E , let $M(P, Q)$ be the mid-point of the line segment joining P and Q , and $M(P, Q')$ be the mid-point of the line segment joining P and Q' . Then the maximum possible value of the distance between $M(P, Q)$ and $M(P, Q')$, as P , Q and Q' vary on E , is _____.

Answer (4)

Sol. Let $P(\alpha)$, $Q(\theta)$, $Q'(\theta')$

$$M = \frac{1}{2}(4 \cos \alpha + 4 \cos \theta), \frac{1}{2}(3 \sin \alpha + 3 \sin \theta)$$

$$M' = \frac{1}{2}(4 \cos \alpha + 4 \cos \theta'), \frac{1}{2}(3 \sin \alpha + 3 \sin \theta')$$

$$MM' = \frac{1}{2} \sqrt{(4 \cos \theta - 4 \cos \theta')^2 + (3 \sin \theta - 3 \sin \theta')^2}$$

$$MM' = \frac{1}{2} \text{ distance between } Q \text{ and } Q'$$

MM' is not depending on P

Maximum of QQ' is possible when $QQ' =$ major axis

$$QQ' = 2(4) = 8$$

$$MM' = \frac{1}{2} \cdot (QQ')$$

$$MM' = 4$$

19. For any real number x , let $[x]$ denote the largest integer less than or equal to x . If $I = \int_0^{10} \left[\sqrt{\frac{10x}{x+1}} \right] dx$, then the value of $9I$ is _____.

Answer (182.00)

Sol. $I = \int_0^{10} \left[\sqrt{\frac{10x}{x+1}} \right] dx$

$$y = \frac{10x}{x+1}, \quad 0 \leq x \leq 10$$

$$xy + y = 10x$$

$$x = \frac{y}{10-y}$$

$$0 \leq \frac{y}{10-y} \leq 10$$

$$\frac{y}{10-y} \geq 0 \quad \text{and} \quad \frac{y}{10-y} - 10 \leq 0$$

$$\frac{y}{y-10} \leq 0 \quad \text{and} \quad \frac{11y-100}{y-10} \geq 0$$

$$\begin{array}{c} + \quad - \quad + \\ \bullet \quad \circ \quad \bullet \\ 0 \quad 10 \end{array} \quad \text{and} \quad \begin{array}{c} + \quad - \quad + \\ \bullet \quad \circ \quad \bullet \\ \frac{100}{11} \quad 10 \end{array}$$

$$y \in [0, 10) \quad \text{and} \quad y \in \left(-\infty, \frac{100}{11} \right] \cup (10, \infty)$$

$$y \in \left[0, \frac{100}{11} \right]$$

$$\sqrt{y} \in \left[0, \frac{10}{\sqrt{11}} \right] \Rightarrow [\sqrt{y}] = \{0, 1, 2, 3\}$$

Case I : $0 \leq \frac{10x}{x+1} < 1$

$$\frac{10x}{x+1} \geq 0 \quad \text{and} \quad \frac{10x}{x+1} - 1 < 0$$

$$\begin{array}{c} + \quad - \quad + \\ \bullet \quad \bullet \quad \bullet \\ -1 \quad 0 \end{array} \quad \text{and} \quad \frac{9x-1}{x+1} < 0$$

$$\begin{array}{c} + \quad - \quad + \\ \bullet \quad \bullet \quad \bullet \\ -1 \quad 1 \\ \frac{1}{9} \end{array}$$

$$x \in (-\infty, -1) \cup [0, \infty) \quad \text{and} \quad x \in \left(-1, \frac{1}{9} \right)$$

$$x \in \left[0, \frac{1}{9} \right) \quad \text{then} \quad \left[\sqrt{\frac{10x}{x+1}} \right] = 0$$

Case II : $1 \leq \frac{10x}{x+1} < 4$

$$\frac{10x}{x+1} - 1 \geq 0 \quad \text{and} \quad \frac{10x}{x+1} - 4 < 0$$

$$\frac{9x-1}{x+1} \geq 0 \quad \text{and} \quad \frac{6x-4}{x+1} < 0$$

$$\begin{array}{c} + \quad - \quad + \\ \circ \quad \bullet \\ -1 \quad \frac{1}{9} \end{array} \quad \text{and} \quad \begin{array}{c} + \quad - \quad + \\ \circ \quad \bullet \quad \circ \\ -1 \quad \frac{2}{3} \end{array}$$

$$x \in (-\infty, -1) \cup \left[\frac{1}{9}, \infty\right) \quad \text{and} \quad x \in \left(-1, \frac{2}{3}\right)$$

$$x \in \left[\frac{1}{9}, \frac{2}{3}\right) \quad , \quad \left\lceil \sqrt{\frac{10x}{x+1}} \right\rceil = 1$$

Case III : $4 \leq \frac{10x}{x+1} < 9$

$$\frac{10x}{x+1} - 4 \geq 0 \quad \text{and} \quad \frac{10x}{x+1} < 9$$

$$\frac{6x-4}{x+1} \geq 0 \quad \text{and} \quad \frac{x-9}{x+1} < 0$$

$$\begin{array}{c} + \quad - \quad + \\ \circ \quad \bullet \\ -1 \quad \frac{2}{3} \end{array} \quad \text{and} \quad \begin{array}{c} + \quad - \quad + \\ \circ \quad \bullet \quad \circ \\ -1 \quad 9 \end{array}$$

$$x \in (-\infty, -1) \cup \left[\frac{2}{3}, \infty\right) \quad x \in (-1, 9)$$

$$x \in \left[\frac{2}{3}, 9\right) \quad ; \quad \left\lceil \sqrt{\frac{10x}{x+1}} \right\rceil = 2$$

Case IV : $x \in [9, 10] \Rightarrow \left\lceil \sqrt{\frac{10x}{x+1}} \right\rceil = 3$

$$I = \int_0^{\frac{1}{9}} 0 \cdot dx + \int_{\frac{1}{9}}^{\frac{2}{3}} 1 \cdot dx + \int_{\frac{2}{3}}^9 2 \cdot dx + \int_9^{10} 3 \cdot dx$$

$$I = \left(\frac{2}{3} - \frac{1}{9}\right) + 2\left(9 - \frac{2}{3}\right) + 3(10 - 9)$$

$$I = \frac{5}{9} + \frac{50}{3} + 3$$

$$9I = 182$$