

JEE(Advanced) – 2018 TEST PAPER - 2 WITH SOLUTION

(Exam Date: 20-05-2018)

PART-1 : MATHEMATICS

SECTION 1

1. For any positive integer n , define $f_n : (0, \infty) \rightarrow \mathbb{R}$ as

$$f_n(x) = \sum_{j=1}^n \tan^{-1} \left(\frac{1}{1+(x+j)(x+j-1)} \right) \text{ for all } x \in (0, \infty).$$

(Here, the inverse trigonometric function $\tan^{-1}x$ assume values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.)

Then, which of the following statement(s) is (are) TRUE ?

(A) $\sum_{j=1}^5 \tan^2(f_j(0)) = 55$

(B) $\sum_{j=1}^{10} (1 + f'_j(0)) \sec^2(f_j(0)) = 10$

(C) For any fixed positive integer n , $\lim_{x \rightarrow \infty} \tan(f_n(x)) = \frac{1}{n}$

(D) For any fixed positive integer n , $\lim_{x \rightarrow \infty} \sec^2(f_n(x)) = 1$

Ans. (D)

Sol. $f_n(x) = \sum_{j=1}^n \tan^{-1} \left(\frac{(x+j) - (x+j-1)}{1+(x+j)(x+j-1)} \right)$

$$f_n(x) = \sum_{j=1}^n [\tan^{-1}(x+j) - \tan^{-1}(x+j-1)]$$

$$f_n(x) = \tan^{-1}(x+n) - \tan^{-1}x$$

$$\therefore \tan(f_n(x)) = \tan[\tan^{-1}(x+n) - \tan^{-1}x]$$

$$\tan(f_n(x)) = \frac{(x+n) - x}{1+x(x+n)}$$

$$\tan(f_n(x)) = \frac{n}{1+x^2+nx}$$

$$\therefore \sec^2(f_n(x)) = 1 + \tan^2(f_n(x))$$

$$\sec^2(f_n(x)) = 1 + \left(\frac{n}{1+x^2+nx} \right)^2$$

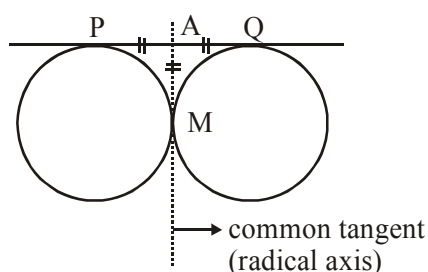
$$\lim_{x \rightarrow \infty} \sec^2(f_n(x)) = \lim_{x \rightarrow \infty} 1 + \left(\frac{n}{1+x^2+nx} \right)^2 = 1$$

2. Let T be the line passing through the points $P(-2, 7)$ and $Q(2, -5)$. Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangent to S_1 at P and tangent to S_2 at Q , and also such that S_1 and S_2 touch each other at a point, say, M . Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1 . Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point $R(1, 1)$ be F_2 . Let E_2 be the set of the mid-points of the line segments in the set F_2 . Then, which of the following statement(s) is (are) TRUE ?

- (A) The point $(-2, 7)$ lies in E_1
- (B) The point $\left(\frac{4}{5}, \frac{7}{5}\right)$ does **NOT** lie in E_2
- (C) The point $\left(\frac{1}{2}, 1\right)$ lies in E_2
- (D) The point $\left(0, \frac{3}{2}\right)$ does **NOT** lie in E_1

Ans. (D)

Sol.



$$AP = AQ = AM$$

Locus of M is a circle having PQ as its diameter

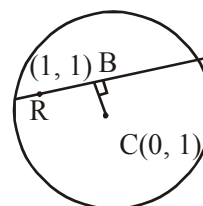
$$\text{Hence, } E_1 : (x-2)(x+2) + (y-7)(y+5) = 0 \text{ and } x \neq \pm 2$$

Locus of B (midpoint)

is a circle having RC as its diameter

$$E_2 : x(x-1) + (y-1)^2 = 0$$

Now, after checking the options, we get (D)



3. Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3 \in \mathbb{R}$ and the system of equations (in real variables)

$$\begin{aligned} -x + 2y + 5z &= b_1 \\ 2x - 4y + 3z &= b_2 \\ x - 2y + 2z &= b_3 \end{aligned}$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one

solution of each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$?

- (A) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$
 (B) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$
 (C) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$
 (D) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$

Ans. (A,D)

Sol. We find $D = 0$ & since no pair of planes are parallel, so there are infinite number of solutions.

$$\text{Let } \alpha P_1 + \lambda P_2 = P_3$$

$$\Rightarrow P_1 + 7P_2 = 13P_3$$

$$\Rightarrow b_1 + 7b_2 = 13b_3$$

(A) $D \neq 0 \Rightarrow$ unique solution for any b_1, b_2, b_3

(B) $D = 0$ but $P_1 + 7P_2 \neq 13P_3$

(C) As planes are parallel and there exist infinite ordered triplet for which they will be non coincident although satisfying $b_1 + 7b_2 = 13b_3$.

$\therefore$ rejected.

(D) $D \neq 0$

4. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin $O(0, 0)$ and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then the which of the following statement(s) is (are) TRUE ?

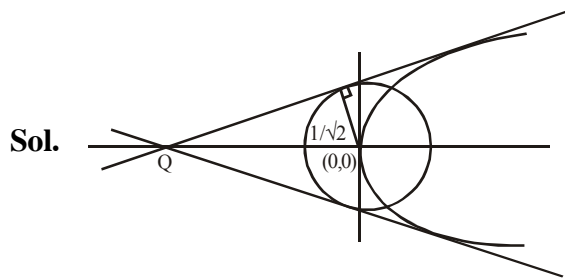
(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1

(B) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(C) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{4\sqrt{2}}(\pi - 2)$

(D) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{16}(\pi - 2)$

Ans. (A,C)



Let equation of common tangent is $y = mx + \frac{1}{m}$

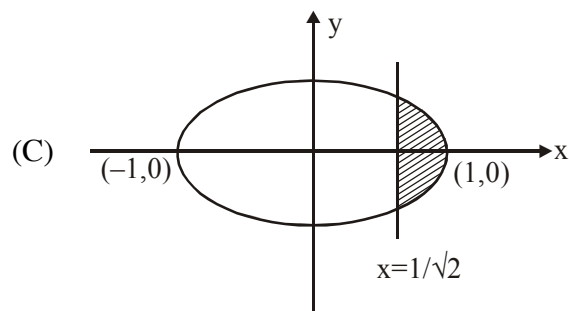
$$\therefore \left| \frac{0+0+\frac{1}{m}}{\sqrt{1+m^2}} \right| = \frac{1}{\sqrt{2}} \Rightarrow m^4 + m^2 - 2 = 0 \Rightarrow m = \pm 1$$

Equation of common tangents are $y = x + 1$ and $y = -x - 1$

point Q is $(-1, 0)$

$\therefore$ Equation of ellipse is $\frac{x^2}{1} + \frac{y^2}{1/2} = 1$

$$(A) \quad e = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}} \quad \text{and} \quad LR = \frac{2b^2}{a} = 1$$



$$\text{Area} = 2 \cdot \int_{1/\sqrt{2}}^1 \frac{1}{\sqrt{2}} \cdot \sqrt{1-x^2} dx = \sqrt{2} \left[\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x \right]_{1/\sqrt{2}}^1$$

$$= \sqrt{2} \left[\frac{\pi}{4} - \left(\frac{1}{4} + \frac{\pi}{8} \right) \right] = \sqrt{2} \left(\frac{\pi}{8} - \frac{1}{4} \right) = \frac{\pi-2}{4\sqrt{2}}$$

correct answer are (A) and (D)

5. Let s, t, r be the non-zero complex numbers and L be the set of solutions $z = x + iy$ ($x, y \in \mathbb{R}, i = \sqrt{-1}$) of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE ?
- (A) If L has exactly one element, then $|s| \neq |t|$
- (B) If $|s| = |t|$, then L has infinitely many elements
- (C) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2
- (D) If L has more than one element, then L has infinitely many elements

Ans. (A,C,D)

Sol. Given

$$sz + t\bar{z} + r = 0 \quad (1)$$

$$\bar{z} = x - iy \text{ (Conjugate of } z\text{)}$$

$$\text{Taking conjugate throughout } \bar{s}\bar{z} + \bar{t}z + \bar{r} = 0 \quad (2)$$

Adding (1) and (2)

$$(s + \bar{t})z + (\bar{s} + t)\bar{z} + (r + \bar{r}) = 0$$

And Subtracting (1) and (2)

$$(s - \bar{t})z + (t - \bar{s})\bar{z} + (r - \bar{r}) = 0$$

For unique solution

$$\frac{t + \bar{s}}{t - s} \neq \frac{s + \bar{t}}{s - \bar{t}}$$

On further simplification $\Rightarrow |t| \neq |s|$

Hence option A proved.

If the lines coincide, then

$$\frac{t + \bar{s}}{t - s} = \frac{\bar{t} + s}{s - \bar{t}} = \frac{r + \bar{r}}{r - \bar{r}}$$

On comparing

$$\frac{t + \bar{s}}{t - s} = \frac{r + \bar{r}}{r - \bar{r}}$$

and simplification, we get $\Rightarrow |s| = |t|$

The lines can be parallel or coincidental.

Since, no concrete outcome.

Hence, option B is not correct.

Clearly L is either a single or represents a line and $|z - 1 + i| = 5$ represents a circle.

$\therefore$ Intersection of L and $\{|z - 1 + i| = 5\}$ is ATMOST 2.

Hence, option C is correct.

Let $s = \alpha_1 + i\beta_1$; $t = \alpha_2 + i\beta_2$ and $r = \alpha_3 + i\beta_3$

Then $sz + t\bar{z} + r = 0$

$$\Rightarrow (\alpha_1 + \alpha_2)x + (\beta_2 - \beta_1)y + \alpha_3 = 0$$

$$\text{and } (\beta_1 + \beta_2)x + (\alpha_1 - \alpha_2)y + \beta_3 = 0$$

If L has more than 1 element then it implies L will have ∞ elements.

As L represents linear equation in x and y.

Hence, option D is correct.

6. Let $f : (0, \pi) \rightarrow \mathbb{R}$ be a twice differentiable function such that

$$\lim_{t \rightarrow x} \frac{f(x)\sin t - f(t)\sin x}{t - x} = \sin^2 x \text{ for all } x \in (0, \pi).$$

If $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$, then which of the following statement(s) is (are) TRUE ?

(A) $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$

(B) $f(x) < \frac{x^4}{6} - x^2$ for all $x \in (0, \pi)$

(C) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$

(D) $f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

Ans. (B,C,D)

Sol. $\lim_{t \rightarrow x} \frac{f(x) \sin t - f(t) \sin x}{t - x} = \sin^2 x$

by using L'Hopital

$$\lim_{t \rightarrow x} \frac{f(x) \cos t - f'(t) \sin x}{1} = \sin^2 x$$

$$\Rightarrow f(x) \cos x - f'(x) \sin x = \sin^2 x$$

$$\Rightarrow - \left(\frac{f'(x) \sin x - f(x) \cos x}{\sin^2 x} \right) = 1$$

$$\Rightarrow -d \left(\frac{f(x)}{\sin x} \right) = 1$$

$$\Rightarrow \frac{f(x)}{\sin x} = -x + c$$

Put $x = \frac{\pi}{6}$ & $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$

$$\therefore c = 0 \Rightarrow f(x) = -x \sin x$$

(A) $f\left(\frac{\pi}{4}\right) = \frac{-\pi}{4} \frac{1}{\sqrt{2}}$

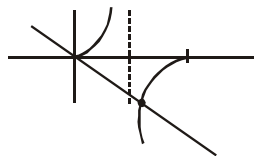
(B) $f(x) = -x \sin x$

$$\text{as } \sin x > x - \frac{x^3}{6}, -x \sin x < -x^2 + \frac{x^4}{6}$$

$$\therefore f(x) < -x^2 + \frac{x^4}{6} \quad \forall x \in (0, \pi)$$

(C) $f'(x) = -\sin x - x \cos x$

$$f'(x) = 0 \Rightarrow \tan x = -x \Rightarrow \text{there exist } \alpha \in (0, \pi) \text{ for which } f'(\alpha) = 0$$



(D) $f''(x) = -2\cos x + x \sin x$

$$f''\left(\frac{\pi}{2}\right) = \frac{\pi}{2}, \quad f''\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$$

$$f''\left(\frac{\pi}{2}\right) + f''\left(\frac{\pi}{2}\right) = 0$$

SECTION 2

7. The value of the integral

$$\int_0^{\frac{1}{2}} \frac{1 + \sqrt{3}}{\left((x+1)^2(1-x)^6\right)^{\frac{1}{4}}} dx$$

is _____ .

Ans. (2)

Sol.
$$\int_0^{\frac{1}{2}} \frac{(1 + \sqrt{3}) dx}{\left[(1+x)^2(1-x)^6\right]^{\frac{1}{4}}}$$

$$\int_0^{\frac{1}{2}} \frac{(1 + \sqrt{3}) dx}{(1+x)^2 \left[\frac{(1-x)^6}{(1+x)^6} \right]^{\frac{1}{4}}}$$

Put $\frac{1-x}{1+x} = t \Rightarrow \frac{-2dx}{(1+x)^2} = dt$

$$I = \int_1^{1/3} \frac{(1 + \sqrt{3}) dt}{-2t^{6/4}} = \frac{-(1 + \sqrt{3})}{2} \times \left| \frac{-2}{\sqrt{t}} \right|_1^{1/3} = (1 + \sqrt{3})(\sqrt{3} - 1) = 2$$

8. Let P be a matrix of order 3×3 such that all the entries in P are from the set $\{-1, 0, 1\}$. Then, the maximum possible value of the determinant of P is _____ .

Ans. (4)

Sol.
$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \underbrace{(a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2)}_x - \underbrace{(a_3 b_2 c_1 + a_2 b_1 c_3 + a_1 b_3 c_2)}_y$$

Now if $x \leq 3$ and $y \geq -3$

the Δ can be maximum 6

But it is not possible

as $x = 3 \Rightarrow$ each term of $x = 1$

and $y = 3 \Rightarrow$ each term of $y = -1$

$$\Rightarrow \prod_{i=1}^3 a_i b_i c_i = 1 \text{ and } \prod_{i=1}^3 a_i b_i c_i = -1$$

which is contradiction

so now next possibility is 4

which is obtained as $\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 1(1+1) - 1(-1-1) + 1(1-1) = 4$

9. Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one-one functions from X to Y and β is the number of onto functions from Y to X, then the value of $\frac{1}{5!}(\beta - \alpha)$ is _____ .

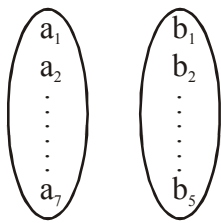
Ans. (119)

Sol. $n(X) = 5$

$n(Y) = 7$

$\alpha \rightarrow$ Number of one-one function $= {}^7C_5 \times 5!$

$\beta \rightarrow$ Number of onto function Y to X



1, 1, 1, 1, 3 1, 1, 1, 2, 2

$$\frac{7!}{3!4!} \times 5! + \frac{7!}{(2!)^3 3!} \times 5! = ({}^7C_3 + 3 \cdot {}^7C_3) 5! = 4 \times {}^7C_3 \times 5!$$

$$\frac{\beta - \alpha}{5!} = 4 \times {}^7C_3 - {}^7C_5 = 4 \times 35 - 21 = 119$$

10. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = (2 + 5y)(5y - 2),$$

then the value of $\lim_{x \rightarrow -\infty} f(x)$ is _____ .

Ans. (0.4)

Sol. $\frac{dy}{dx} = 25y^2 - 4$

So, $\frac{dy}{25y^2 - 4} = dx$

Integrating, $\frac{1}{25} \times \frac{1}{2 \times \frac{2}{5}} \ln \left| \frac{y - \frac{2}{5}}{y + \frac{2}{5}} \right| = x + c$

$$\Rightarrow \ln \left| \frac{5y-2}{5y+2} \right| = 20(x+c)$$

Now, $c = 0$ as $f(0) = 0$

$$\text{Hence } \left| \frac{5y-2}{5y+2} \right| = e^{(20x)}$$

$$\lim_{x \rightarrow -\infty} \left| \frac{5f(x)-2}{5f(x)+2} \right| = \lim_{x \rightarrow -\infty} e^{(20x)}$$

$$\text{Now, RHS} = 0 \Rightarrow \lim_{x \rightarrow -\infty} (5f(x) - 2) = 0$$

$$\Rightarrow \lim_{x \rightarrow -\infty} f(x) = \frac{2}{5}$$

- 11.** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function with $f(0) = 1$ and satisfying the equation

$$f(x+y) = f(x)f'(y) + f'(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

Then, the value of $\log_e(f(4))$ is _____.

Ans. (2)

Sol. $P(x, y) : f(x+y) = f(x)f'(y) + f'(x)f(y) \forall x, y \in \mathbb{R}$

$$P(0, 0) : f(0) = f(0)f'(0) + f'(0)f(0)$$

$$\Rightarrow 1 = 2f'(0)$$

$$\Rightarrow f'(0) = \frac{1}{2}$$

$$P(x, 0) : f(x) = f(x).f'(0) + f'(x).f(0)$$

$$\Rightarrow f(x) = \frac{1}{2}f(x) + f'(x)$$

$$\Rightarrow f'(x) = \frac{1}{2}f(x)$$

$$\Rightarrow f(x) = e^{\frac{1}{2}x}$$

$$\Rightarrow \ln(f(4)) = 2$$

- 12.** Let P be a point in the first octant, whose image Q in the plane $x + y = 3$ (that is, the line segment PQ is perpendicular to the plane $x + y = 3$ and the mid-point of PQ lies in the plane $x + y = 3$) lies on the z-axis. Let the distance of P from the x-axis be 5. If R is the image of P in the xy-plane, then the length of PR is _____.

Ans. (8)

Sol. Let

$$P(\alpha, \beta, \gamma)$$

$$Q(0, 0, \gamma) \quad \&$$

$$R(\alpha, \beta, -\gamma)$$

$$\text{Now, } \overline{PQ} \parallel \hat{i} + \hat{j} \Rightarrow (\alpha\hat{i} + \beta\hat{j}) \parallel (\hat{i} + \hat{j})$$

$$\Rightarrow \alpha = \beta$$

$$\text{Also, mid point of PQ lies on the plane} \Rightarrow \frac{\alpha}{2} + \frac{\beta}{2} = 3 \Rightarrow \alpha + \beta = 6 \Rightarrow \alpha = 3$$

$$\text{Now, distance of point P from X-axis is } \sqrt{\beta^2 + \gamma^2} = 5$$

$$\Rightarrow \beta^2 + \gamma^2 = 25 \Rightarrow \gamma^2 = 16$$

$$\text{as } \beta = \alpha = 3$$

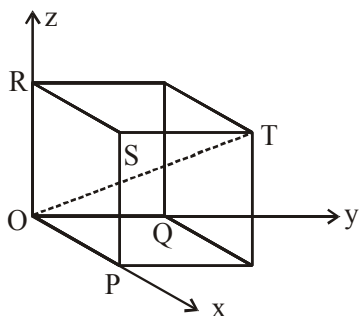
$$\text{as } \gamma = 4$$

$$\text{Hence, PR} = 2\gamma = 8$$

- 13.** Consider the cube in the first octant with sides OP, OQ and OR of length 1, along the x-axis, y-axis and z-axis, respectively, where O(0, 0, 0) is the origin. Let $S\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ be the centre of the cube and T be the vertex of the cube opposite to the origin O such that S lies on the diagonal OT. If $\vec{p} = \overrightarrow{SP}$, $\vec{q} = \overrightarrow{SQ}$, $\vec{r} = \overrightarrow{SR}$ and $\vec{t} = \overrightarrow{ST}$, then the value of $|(\vec{p} \times \vec{q}) \times (\vec{r} \times \vec{t})|$ is _____ .

Ans. (0.5)

Sol.



$$\vec{p} = \overrightarrow{SP} = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2}(\hat{i} - \hat{j} - \hat{k})$$

$$\vec{q} = \overrightarrow{SQ} = \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2}(-\hat{i} + \hat{j} - \hat{k})$$

$$\vec{r} = \overrightarrow{SR} = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}(-\hat{i} - \hat{j} + \hat{k})$$

$$\vec{t} = \overrightarrow{ST} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}(\hat{i} + \hat{j} + \hat{k})$$

$$|(\vec{p} \times \vec{q}) \times (\vec{r} \times \vec{t})| = \frac{1}{4} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{vmatrix} \times \frac{1}{4} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{16} |(2\hat{i} + 2\hat{j}) \times (-2\hat{i} + 2\hat{j})| = \left| \frac{\hat{k}}{2} \right| = \frac{1}{2}$$

- 14.** Let $X = \binom{10}{1}C_1^2 + 2\binom{10}{2}C_2^2 + 3\binom{10}{3}C_3^2 + \dots + 10\binom{10}{10}C_{10}^2$, where ${}^{10}C_r, r \in \{1, 2, \dots, 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430}X$ is _____.

Ans. (646)

Sol. $X = \sum_{r=0}^n r \cdot \binom{n}{r} C_r^2; n = 10$

$$X = n \cdot \sum_{r=0}^n \binom{n}{r} \cdot {}^{n-1}C_{r-1}$$

$$X = n \cdot \sum_{r=1}^n \binom{n}{n-r} \cdot {}^{n-1}C_{r-1}$$

$$X = n \cdot {}^{2n-1}C_{n-1}; n = 10$$

$$X = 10 \cdot {}^{19}C_9$$

$$\frac{X}{1430} = \frac{1}{143} \cdot {}^{19}C_9$$

$$= 646$$

SECTION 3

15. Let $E_1 = \left\{ x \in \mathbb{R} : x \neq 1 \text{ and } \frac{x}{x-1} > 0 \right\}$

and $E_2 = \left\{ x \in E_1 : \sin^{-1} \left(\log_e \left(\frac{x}{x-1} \right) \right) \text{ is a real number} \right\}.$

$\left(\text{Here, the inverse trigonometric function } \sin^{-1}x \text{ assumes values in } \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right)$

Let $f : E_1 \rightarrow \mathbb{R}$ be the function defined by $f(x) = \log_e \left(\frac{x}{x-1} \right)$

and $g : E_2 \rightarrow \mathbb{R}$ be the function defined by $g(x) = \sin^{-1} \left(\log_e \left(\frac{x}{x-1} \right) \right).$

LIST-I

P. The range of f is

Q. The range of g contains

R. The domain of f contains

S. The domain of g is

LIST-II

1. $\left(-\infty, \frac{1}{1-e} \right] \cup \left[\frac{e}{e-1}, \infty \right)$

2. $(0, 1)$

3. $\left[-\frac{1}{2}, \frac{1}{2} \right]$

4. $(-\infty, 0) \cup (0, \infty)$

5. $\left(-\infty, \frac{e}{e-1} \right]$

6. $(-\infty, 0) \cup \left(\frac{1}{2}, \frac{e}{e-1} \right]$

The correct option is :

(A) $P \rightarrow 4; Q \rightarrow 2; R \rightarrow 1; S \rightarrow 1$

(B) $P \rightarrow 3; Q \rightarrow 3; R \rightarrow 6; S \rightarrow 5$

(C) $P \rightarrow 4; Q \rightarrow 2; R \rightarrow 1; S \rightarrow 6$

(D) $P \rightarrow 4; Q \rightarrow 3; R \rightarrow 6; S \rightarrow 5$

Ans. (A)

Sol. $E_1: \frac{x}{x-1} > 0$

$$\begin{array}{c} + \quad \quad - \quad \quad + \\ \hline 0 \quad \quad 1 \end{array}$$

$$\Rightarrow E_1: x \in (-\infty, 0) \cup (1, \infty)$$

$$E_2: -1 \leq \ln\left(\frac{x}{x+1}\right) \leq 1$$

$$\frac{1}{e} \leq \frac{x}{x-1} \leq e$$

Now $\frac{x}{x-1} - \frac{1}{e} \geq 0$

$$\Rightarrow \frac{(e-1)x+1}{e(x-1)} \geq 0$$

$$\begin{array}{c} + \quad \quad - \quad \quad + \\ \hline -1/(e-1) \quad \quad 1 \end{array}$$

$$\Rightarrow x \in \left(-\infty, \frac{1}{1-e}\right] \cup (1, \infty)$$

also $\frac{x}{x-1} - e \leq 0$

$$\frac{(e-1)x-e}{x-1} \geq 0$$

$$\begin{array}{c} + \quad \quad - \quad \quad + \\ \hline 1 \quad \quad e/(e-1) \end{array}$$

$$\Rightarrow x \in (-\infty, 1) \cup \left[\frac{e}{e-1}, \infty\right]$$

So $E_2: \left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right]$

as Range of $\frac{x}{x-1}$ is $\mathbb{R}^+ - \{1\}$

$$\Rightarrow \text{Range of } f \text{ is } \mathbb{R} - \{0\} \text{ or } (-\infty, 0) \cup (0, \infty)$$

$$\text{Range of } g \text{ is } \left[\frac{-\pi}{2}, \frac{\pi}{2} \right] \setminus \{0\} \text{ or } \left[-\frac{\pi}{2}, 0 \right) \cup \left(0, \frac{\pi}{2} \right]$$

Now $P \rightarrow 4, Q \rightarrow 2, R \rightarrow 1, S \rightarrow 1$

Hence A is correct

16. In a high school, a committee has to be formed from a group of 6 boys $M_1, M_2, M_3, M_4, M_5, M_6$ and 5 girls G_1, G_2, G_3, G_4, G_5 .

- (i) Let α_1 be the total number of ways in which the committee can be formed such that the committee has 5 members, having exactly 3 boys and 2 girls.
- (ii) Let α_2 be the total number of ways in which the committee can be formed such that the committee has at least 2 members, and having an equal number of boys and girls.
- (iii) Let α_3 be the total number of ways in which the committee can be formed such that the committee has 5 members, at least 2 of them being girls.
- (iv) Let α_4 be the total number of ways in which the committee can be formed such that the committee has 4 members, having at least 2 girls and such that both M_1 and G_1 are **NOT** in the committee together.

LIST-I

LIST-II

P. The value of α_1 is **1.** 136

Q. The value of α_2 is **2.** 189

R. The value of α_3 is **3.** 192

S. The value of α_4 is **4.** 200

5. 381

6. 461

The correct option is :-

(A) $P \rightarrow 4; Q \rightarrow 6, R \rightarrow 2; S \rightarrow 1$

(B) $P \rightarrow 1; Q \rightarrow 4; R \rightarrow 2; S \rightarrow 3$

(C) $P \rightarrow 4; Q \rightarrow 6, R \rightarrow 5; S \rightarrow 2$

(D) $P \rightarrow 4; Q \rightarrow 2; R \rightarrow 3; S \rightarrow 1$

Ans. (C)

Sol. (1) $\alpha_1 = \binom{6}{3} \binom{5}{2} = 200$

So $P \rightarrow 4$

$$(2) \quad \alpha_2 = \binom{6}{1}\binom{5}{1} + \binom{6}{2}\binom{5}{2} + \binom{6}{3}\binom{5}{3} + \binom{6}{4}\binom{5}{4} + \binom{6}{5}\binom{5}{5}$$

$$= \binom{11}{5} - 1$$

$$= 46!$$

$$\text{So } Q \rightarrow 6$$

$$(3) \quad \alpha_3 = \binom{5}{2}\binom{6}{3} + \binom{5}{3}\binom{6}{2} + \binom{5}{4}\binom{6}{1} + \binom{5}{5}\binom{6}{0}$$

$$= \binom{11}{5} - \binom{5}{0}\binom{6}{5} - \binom{5}{1}\binom{6}{4}$$

$$= 381$$

$$\text{So } R \rightarrow 5$$

$$(4) \quad \alpha_2 = \binom{5}{2}\binom{6}{2} - \binom{4}{1}\binom{5}{1} + \binom{5}{3}\binom{6}{1} - \binom{4}{2}\binom{1}{1} + \binom{5}{4} = 189$$

$$\text{So } S \rightarrow 2$$

17. Let $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$, be a hyperbola in the xy -plane whose conjugate axis LM subtends

an angle of 60° at one of its vertices N . Let the area of the triangle LMN be $4\sqrt{3}$.

LIST-I

LIST-II

P. The length of the conjugate axis of H is

1. 8

Q. The eccentricity of H is

2. $\frac{4}{\sqrt{3}}$

R. The distance between the foci of H is

3. $\frac{2}{\sqrt{3}}$

S. The length of the latus rectum of H is

4. 4

The correct option is :

(A) **P** $\rightarrow$ 4; **Q** $\rightarrow$ 2; **R** $\rightarrow$ 1; **S** $\rightarrow$ 3

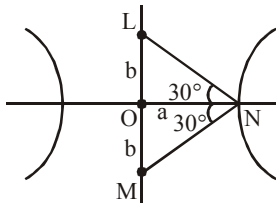
(B) **P** $\rightarrow$ 4; **Q** $\rightarrow$ 3; **R** $\rightarrow$ 1; **S** $\rightarrow$ 2

(C) **P** $\rightarrow$ 4; **Q** $\rightarrow$ 1; **R** $\rightarrow$ 3; **S** $\rightarrow$ 2

(D) **P** $\rightarrow$ 3; **Q** $\rightarrow$ 4; **R** $\rightarrow$ 2; **S** $\rightarrow$ 1

Ans. (B)

Sol.



$$\tan 30^\circ = \frac{b}{a}$$

$$\Rightarrow a = b\sqrt{3}$$

$$\text{Now area of } \triangle LMN = \frac{1}{2} \cdot 2b \cdot b\sqrt{3}$$

$$4\sqrt{3} = \sqrt{3}b^2$$

$$\Rightarrow b = 2 \quad \& \quad a = 2\sqrt{3}$$

$$\Rightarrow e = \sqrt{1 + \frac{b^2}{a^2}} = \frac{2}{\sqrt{3}}$$

P. Length of conjugate axis = $2b = 4$

So P $\rightarrow 4$

Q. Eccentricity $e = \frac{2}{\sqrt{3}}$

So Q $\rightarrow 3$

R. Distance between foci = $2ae$

$$= 2(2\sqrt{3})\left(\frac{2}{\sqrt{3}}\right) = 8$$

So R $\rightarrow 1$

S. Length of latus rectum = $\frac{2b^2}{a} = \frac{2(2)^2}{2\sqrt{3}} = \frac{4}{\sqrt{3}}$

So S $\rightarrow 2$

18. Let $f_1 : \mathbb{R} \rightarrow \mathbb{R}$, $f_2 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$, $f_3 : \left(-1, e^{\frac{\pi}{2}} - 2\right) \rightarrow \mathbb{R}$ and $f_4 : \mathbb{R} \rightarrow \mathbb{R}$ be functions defined

by

$$(i) f_1(x) = \sin\left(\sqrt{1-e^{-x^2}}\right)$$

$$(ii) f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1} x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}, \text{ where the inverse trigonometric function } \tan^{-1} x \text{ assumes values}$$

$$\text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

$$(iii) f_3(x) = [\sin(\log_e(x+2))], \text{ where for } t \in \mathbb{R}, [t] \text{ denotes the greatest integer less than or equal to } t,$$

$$(iv) f_4(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

List-I

P. the function f_1 is

Q. The function f_2 is

R. The function f_3 is

S. The function f_4 is

List-II

1. NOT continuous at $x = 0$

2. continuous at $x = 0$ and NOT differentiable at $x = 0$

3. differentiable at $x = 0$ and its derivative is NOT continuous at $x = 0$

4. differentiable at $x = 0$ and its derivative is continuous at $x = 0$

The correct option is :

(A) **P** $\rightarrow$ 2; **Q** $\rightarrow$ 3; **R** $\rightarrow$ 1; **S** $\rightarrow$ 4

(B) **P** $\rightarrow$ 4; **Q** $\rightarrow$ 1; **R** $\rightarrow$ 2; **S** $\rightarrow$ 3

(C) **P** $\rightarrow$ 4; **Q** $\rightarrow$ 2; **R** $\rightarrow$ 1; **S** $\rightarrow$ 3

(D) **P** $\rightarrow$ 2; **Q** $\rightarrow$ 1; **R** $\rightarrow$ 4; **S** $\rightarrow$ 3

Ans. (D)

Sol. (i) $f(x) = \sin \sqrt{1 - e^{-x^2}}$

$$f_1'(x) = \cos \sqrt{1 - e^{-x^2}} \cdot \frac{1}{2\sqrt{1 - e^{-x^2}}} (0 - e^{-x^2} \cdot (-2x))$$

at $x = 0$ $f_1'(x)$ does not exist

So. P $\rightarrow$ 2

$$(ii) f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1} x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x} \cdot \frac{x}{\tan^{-1} x} = 1$$

$\Rightarrow f_2(x)$ does not continuous at $x = 0$

So Q $\rightarrow$ 1

$$(iii) f_3(x) = [\sin \ell n(x+2)] = 0$$

$$1 < x + 2 < e^{\pi/2}$$

$$\Rightarrow 0 < \ell n(x+2) < \frac{\pi}{2}$$

$$\Rightarrow 0 < \sin(\ell n(x+2)) < 1$$

$$\Rightarrow f_3(x) = 0$$

So R $\rightarrow$ 4

$$(iv) f_4(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$$

So S $\rightarrow$ 3