

JEE(Advanced) – 2017 TEST PAPER WITH SOLUTION

(HELD ON SUNDAY 21st MAY, 2017)

MATHEMATICS

SECTION-I : (Maximum Marks : 21)

- This section contains **SEVEN** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is correct.
- For each question, darken the bubble corresponding to the correct option in the ORS.
- For each question, marks will be awarded in one of the following categories :

Full Marks : +3 If only the bubble corresponding to the correct option is darkened.

Zero Marks : 0 If none of the bubbles is darkened.

Negative Marks : -1 In all other cases

37. Three randomly chosen nonnegative integers x , y and z are found to satisfy the equation $x + y + z = 10$. Then the probability that z is even, is

(A) $\frac{36}{55}$ (B) $\frac{6}{11}$ (C) $\frac{5}{11}$ (D) $\frac{1}{2}$

Ans. (B)

Sol. Let $z = 2k$, where $k = 0, 1, 2, 3, 4, 5$

$$\therefore x + y = 10 - 2k$$

Number of non negative integral solutions

$$\sum_{k=0}^5 {}^{11-2k}C_1 = \sum 11-2k = 36$$

$$\text{Total cases} = {}^{10+3-1}C_{3-1} = 66$$

$$\text{Reqd. prob.} = \frac{36}{66} = \frac{6}{11}$$

38. Let $S = \{1, 2, 3, \dots, 9\}$. For $k = 1, 2, \dots, 5$, let N_k be the number of subsets of S , each containing five elements out of which exactly k are odd. Then $N_1 + N_2 + N_3 + N_4 + N_5 =$

(A) 125 (B) 252 (C) 210 (D) 126

Ans. (D)

Sol. $N_1 + N_2 + N_3 + N_4 + N_5 = \text{Total ways} - \{\text{when no odd}\}$

$$\text{Total ways} = {}^9C_5$$

Number of ways when no odd, is zero ($\because$ only available even are 2, 4, 6, 8)

$$\therefore \text{Ans : } {}^9C_5 - \text{zero} = 126$$

39. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a twice differentiable function such that $f''(x) > 0$ for all $x \in \mathbb{R}$, and $f\left(\frac{1}{2}\right) = \frac{1}{2}$, $f(1) = 1$, then

(A) $0 < f'(1) \leq \frac{1}{2}$ (B) $f'(1) \leq 0$ (C) $f'(1) > 1$ (D) $\frac{1}{2} < f'(1) \leq 1$

Ans. (C)

Sol. Using LMVT on $f(x)$ for $x \in \left[\frac{1}{2}, 1\right]$

$$\frac{f(1) - f\left(\frac{1}{2}\right)}{1 - \frac{1}{2}} = f'(c), \text{ where } c \in \left(\frac{1}{2}, 1\right)$$

$$\frac{1 - \frac{1}{2}}{\frac{1}{2}} = f'(c) \Rightarrow f'(c) = 1, \text{ where } c \in \left(\frac{1}{2}, 1\right)$$

$\therefore f'(x)$ is an increasing function $\forall x \in \mathbb{R}$

$\therefore f'(1) > 1$

40. If $y = y(x)$ satisfies the differential equation

$$8\sqrt{x}(\sqrt{9+\sqrt{x}})dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1} dx, \quad x > 0$$

and $y(0) = \sqrt{7}$, then $y(256) =$

(A) 80

(B) 3

(C) 16

(D) 9

Ans. (B)

Sol. $y = \frac{1}{8} \int \frac{dx}{\sqrt{4+\sqrt{9+x}} \cdot \sqrt{x} \cdot \sqrt{9+\sqrt{x}}}$

put $\sqrt{9+\sqrt{x}} = t \Rightarrow \frac{dx}{\sqrt{x} \cdot \sqrt{9+\sqrt{x}}} = 4dt$

$$\therefore y = \frac{4}{8} \int \frac{dt}{\sqrt{4+t}}$$

$$\Rightarrow y = \sqrt{4+t} + C$$

$$\Rightarrow y(x) = \sqrt{4+\sqrt{9+\sqrt{x}}} + C$$

$$\text{at } x = 0 : y(0) = \sqrt{7} \Rightarrow C = 0$$

$$\therefore y(x) = \sqrt{4+\sqrt{9+\sqrt{x}}}$$

$$\Rightarrow y(256) = 3$$

41. How many 3×3 matrices M with entries from $\{0,1,2\}$ are there, for which the sum of the diagonal entries of $M^T M$ is 5 ?

(A) 198

(B) 126

(C) 135

(D) 162

Ans. (A)

Sol. Let $M = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$$\therefore \text{tr}(M^T M) = a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 + i^2 = 5, \text{ where entries are } \{0,1,2\}$$

Only two cases are possible.

(I) five entries 1 and other four zero

$$\therefore {}^9C_5 \times 1$$

(II) One entry is 2, one entry is 1 and others are 0.

$$\therefore {}^9C_2 \times 2!$$

$$\text{Total} = 126 + 72 = 198.$$

42. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that

$$\overrightarrow{OP} \cdot \overrightarrow{OQ} + \overrightarrow{OR} \cdot \overrightarrow{OS} = \overrightarrow{OR} \cdot \overrightarrow{OP} + \overrightarrow{OQ} \cdot \overrightarrow{OS} = \overrightarrow{OQ} \cdot \overrightarrow{OR} + \overrightarrow{OP} \cdot \overrightarrow{OS}$$

Then the triangle PQR has S as its

(A) incentre (B) orthocenter (C) circumcentre (D) centroid

Ans. (B)

Sol. Let position vector of P($\vec{p}$), Q($\vec{q}$), R($\vec{r}$) & S($\vec{s}$) with respect to O($\vec{o}$)

$$\text{Now, } \overrightarrow{OP} \cdot \overrightarrow{OQ} + \overrightarrow{OR} \cdot \overrightarrow{OS} = \overrightarrow{OR} \cdot \overrightarrow{OP} + \overrightarrow{OQ} \cdot \overrightarrow{OS}$$

$$\Rightarrow \vec{p} \cdot \vec{q} + \vec{r} \cdot \vec{s} = \vec{r} \cdot \vec{p} + \vec{q} \cdot \vec{s}$$

$$\Rightarrow (\vec{p} - \vec{s}) \cdot (\vec{q} - \vec{r}) = 0 \quad \dots(1)$$

$$\text{Also, } \overrightarrow{OR} \cdot \overrightarrow{OP} + \overrightarrow{OQ} \cdot \overrightarrow{OS} = \overrightarrow{OQ} \cdot \overrightarrow{OR} + \overrightarrow{OP} \cdot \overrightarrow{OS}$$

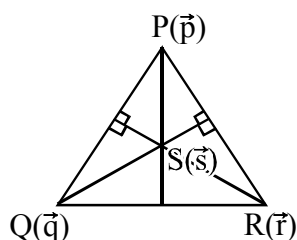
$$\Rightarrow \vec{r} \cdot \vec{p} + \vec{q} \cdot \vec{s} = \vec{q} \cdot \vec{r} + \vec{p} \cdot \vec{s}$$

$$\Rightarrow (\vec{r} - \vec{s}) \cdot (\vec{p} - \vec{q}) = 0 \quad \dots(2)$$

$$\text{Also, } \overrightarrow{OP} \cdot \overrightarrow{OQ} + \overrightarrow{OR} \cdot \overrightarrow{OS} = \overrightarrow{OQ} \cdot \overrightarrow{OR} + \overrightarrow{OP} \cdot \overrightarrow{OS}$$

$$\Rightarrow \vec{p} \cdot \vec{q} + \vec{r} \cdot \vec{s} = \vec{q} \cdot \vec{r} + \vec{p} \cdot \vec{s}$$

$$\Rightarrow (\vec{q} - \vec{s}) \cdot (\vec{p} - \vec{r}) = 0 \quad \dots(3)$$



$\Rightarrow$ Triangle PQR has S as its orthocentre

$\therefore$ option (B) is correct.

43. The equation of the plane passing through the point (1,1,1) and perpendicular to the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$, is-

- (A) $14x + 2y + 15z = 31$ (B) $14x + 2y - 15z = 1$
 (C) $-14x + 2y + 15z = 3$ (D) $14x - 2y + 15z = 27$

Ans. (A)

Sol. The normal vector of required plane is parallel to vector

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 3 & -6 & -2 \end{vmatrix} = -14\hat{i} - 2\hat{j} - 15\hat{k}$$

∴ The equation of required plane passing through (1, 1, 1) will be

$$-14(x - 1) - 2(y - 1) - 15(z - 1) = 0$$

$$\Rightarrow \boxed{14x + 2y + 15z = 31}$$

∴ Option (A) is correct

SECTION-2 : (Maximum Marks : 28)

- This section contains **SEVEN** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four options is (are) correct.
- For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS
- For each question, marks will be awarded in one of the following categories :

Full Marks : +4 If only the bubble(s) corresponding to all the correct option(s) is (are) darkened.

Partial Marks : +1 For darkening a bubble corresponding **to each correct option**, Provided NO incorrect option is darkened.

Zero Marks : 0 If none of the bubbles is darkened.

Negative Marks : -2 In all other cases.

- for example, if (A), (C) and (D) are all the correct options for a question, darkening all these three will get +4 marks; darkening only (A) and (D) will get +2 marks; and darkening (A) and (B) will get -2 marks, as a wrong option is also darkened

44. If $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx$, then

- (A) $I < \frac{49}{50}$ (B) $I < \log_e 99$ (C) $I > \frac{49}{50}$ (D) $I > \log_e 99$

Ans. (B,C)

Sol.
$$I = \sum_{k=1}^{98} \left(\int_k^{k+1} \frac{(k+1)}{x(x+1)} dx \right)$$

$$= \sum_{k=1}^{98} (k+1) \left(\int_k^{k+1} \left(\frac{1}{x} - \frac{1}{x+1} \right) dx \right)$$

$$= \sum_{k=1}^{98} (k+1) \left((\ln x - \ln(x+1))_k^{k+1} \right)$$

$$= \sum_{k=1}^{98} (k+1) (\ln(k+1) - \ln(k+2) - \ln k + \ln(k+1))$$

$$= \sum_{k=1}^{98} ((k+1) \ln(k+1) - k \ln k) - \sum_{k=1}^{98} ((k+1) \ln(k+2) - k \ln(k+1)) + \sum_{k=1}^{98} (\ln(k+1) - \ln k)$$

(Difference series)

$$\therefore I = (99 \ln 99) + (-99 \ln 100 + \ln 2) + (\ln 99) = \ln \left(\frac{2 \times (99)^{100}}{(100)^{99}} \right) \quad \dots\dots(1)$$

For option (B) :

Now, consider $(100)^{99} = (1 + 99)^{99}$

$$= {}^{99}C_0 + {}^{99}C_1(99) + {}^{99}C_2(99)^2 + \dots\dots + {}^{99}C_{97}(99)^{97} + \underbrace{{}^{99}C_{98}(99)^{98}}_{(\text{value}=(99)^{99})} + \underbrace{{}^{99}C_{99}(99)^{99}}_{(\text{value}=(99)^{99})}$$

$$\Rightarrow (100)^{99} > 2.(99)^{99} \Rightarrow \frac{2 \times (99)^{99}}{(100)^{99}} < 1$$

$$\therefore \frac{2 \times (99)^{100}}{(100)^{99}} < 99 \quad (\text{on multiplying by } 99)$$

$$\Rightarrow I < \ln 99$$

For option (C) :

Since,
$$\sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{(x+1)^2} dx < \sum_{k=1}^{98} \int_k^{k+1} \frac{(k+1)dx}{x(x+1)}$$

$$\Rightarrow \sum_{k=1}^{98} \left(\frac{1}{k+2} \right) < I$$

(on integration)

$$\Rightarrow \underbrace{\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots\dots + \frac{1}{100} \right)}_{98 \text{ terms}} < I$$

$$\Rightarrow \frac{98}{100} < \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots\dots + \frac{1}{100} < I$$

$$\therefore I > \frac{49}{50}$$

Hence option (C) is correct.

45. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in \mathbb{R}$, and $f(0) = 1$, then

- (A) $f(x) > e^{2x}$ in $(0, \infty)$ (B) $f(x)$ is decreasing in $(0, \infty)$
 (C) $f(x)$ is increasing in $(0, \infty)$ (D) $f'(x) < e^{2x}$ in $(0, \infty)$

Ans. (A,C)

Sol. Given that,

$$\begin{aligned} f'(x) &> 2f(x) \quad \forall x \in \mathbb{R} \\ \Rightarrow f'(x) - 2f(x) &> 0 \quad \forall x \in \mathbb{R} \\ \therefore e^{-2x} (f'(x) - 2f(x)) &> 0 \quad \forall x \in \mathbb{R} \end{aligned}$$

$$\Rightarrow \frac{d}{dx} (e^{-2x} f(x)) > 0 \quad \forall x \in \mathbb{R}$$

Let $g(x) = e^{-2x} f(x)$

Now, $g'(x) > 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow g(x)$ is strictly increasing $\forall x \in \mathbb{R}$

Also, $g(0) = 1$

$\therefore \forall x > 0$

$\Rightarrow g(x) > g(0) = 1$

$\therefore e^{-2x} \cdot f(x) > 1 \quad \forall x \in (0, \infty) \Rightarrow f(x) > e^{2x} \quad \forall x \in (0, \infty)$

$\therefore$ option (A) is correct

As, $f'(x) > 2f(x) > 2e^{2x} > 2 \quad \forall x \in (0, \infty)$

$\Rightarrow f(x)$ is strictly increasing on $x \in (0, \infty)$

$\Rightarrow$ option (C) is correct

As, we have proved above that

$$f'(x) > 2e^{2x} \quad \forall x \in (0, \infty)$$

$\Rightarrow$ option (D) is incorrect

$\therefore$ options (A) and (C) are correct

46. If $f(x) = \begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$, then

(A) $f'(x) = 0$ at exactly three points in $(-\pi, \pi)$

(B) $f(x)$ attains its maximum at $x = 0$

(C) $f(x)$ attains its minimum at $x = 0$

(D) $f'(x) = 0$ at more than three points in $(-\pi, \pi)$

Ans. (B,D)

Sol. Expansion of determinant

$$f(x) = \cos 2x + \cos 4x$$

$$f'(x) = -2\sin 2x - 4\sin 4x = -2\sin x(1 + 4\cos 2x)$$

$$\begin{array}{c} + \qquad \qquad - \\ \hline \qquad \qquad 0 \end{array} \quad \therefore \text{maxima at } x = 0$$

$$f'(x) = 0 \Rightarrow x = \frac{n\pi}{2}, \cos 2x = -\frac{1}{4}$$

$\Rightarrow$ more than two solutions

47. Let α and β be nonzero real numbers such that $2(\cos\beta - \cos\alpha) + \cos\alpha \cos\beta = 1$. Then which of the following is/are true ?

(A) $\tan\left(\frac{\alpha}{2}\right) - \sqrt{3} \tan\left(\frac{\beta}{2}\right) = 0$

(B) $\sqrt{3} \tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$

(C) $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3} \tan\left(\frac{\beta}{2}\right) = 0$

(D) $\sqrt{3} \tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 0$

Ans. (A,C)

Sol. $2(\cos\beta - \cos\alpha) + \cos\alpha \cos\beta - 1 = 0 \longrightarrow (1)$

$$\text{use } \cos\beta = \frac{1 - \tan^2 \frac{\beta}{2}}{1 + \tan^2 \frac{\beta}{2}} \text{ and } \cos\alpha = \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} \text{ in (1)}$$

$$\text{We get } \tan^2\left(\frac{\alpha}{2}\right) = 3\tan^2\left(\frac{\beta}{2}\right) \Rightarrow \tan\left(\frac{\alpha}{2}\right) + \sqrt{3} \tan\frac{\beta}{2} = 0 \text{ or } \tan\left(\frac{\alpha}{2}\right) - \sqrt{3} \tan\left(\frac{\beta}{2}\right) = 0$$

Hence (A, C)

48. If $g(x) = \int_{\sin x}^{\sin(2x)} \sin^{-1}(t) dt$, then

(A) $g'\left(\frac{\pi}{2}\right) = -2\pi$ (B) $g'\left(-\frac{\pi}{2}\right) = 2\pi$ (C) $g'\left(\frac{\pi}{2}\right) = 2\pi$ (D) $g'\left(-\frac{\pi}{2}\right) = -2\pi$

Ans. (BONUS)

Sol. $g(x) = \int_{\sin x}^{\sin 2x} \sin^{-1} t dt \Rightarrow g'(x) = 2\sin^{-1}(\sin 2x) \times \cos 2x - \sin^{-1}(\sin x) \cos x$

$$\Rightarrow g'\left(\frac{\pi}{2}\right) = 0 \text{ \& } g'\left(-\frac{\pi}{2}\right) = 0$$

No option matches the result

$\Rightarrow$ BONUS

49. If the line $x = \alpha$ divides the area of region $R = \{(x, y) \in \mathbb{R}^2 : x^3 \leq y \leq x, 0 \leq x \leq 1\}$ into two equal parts, then

(A) $\frac{1}{2} < \alpha < 1$ (B) $\alpha^4 + 4\alpha^2 - 1 = 0$ (C) $0 < \alpha \leq \frac{1}{2}$ (D) $2\alpha^4 - 4\alpha^2 + 1 = 0$

Ans. (A,D)

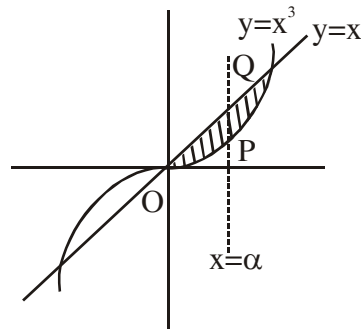
Sol. Area between $y = x^3$ and $y = x$ in $x \in (0,1)$ is

$$A = \int_0^1 (x - x^3) dx = \frac{1}{4}$$

Area of curve linear triangle OPQ $= \frac{A}{2} = \frac{1}{8}$

$$\Rightarrow \int_0^\alpha (x - x^3) dx = \frac{1}{8} \Rightarrow 2\alpha^4 - 4\alpha^2 + 1 = 0$$

$$\Rightarrow (\alpha^2 - 1)^2 = \frac{1}{2} \Rightarrow \alpha^2 = \frac{\sqrt{2} - 1}{\sqrt{2}}$$



50. Let $f(x) = \frac{1-x(1+|1-x|)}{|1-x|} \cos\left(\frac{1}{1-x}\right)$ for $x \neq 1$. Then

(A) $\lim_{x \rightarrow 1^+} f(x)$ does not exist

(B) $\lim_{x \rightarrow 1^-} f(x)$ does not exist

(C) $\lim_{x \rightarrow 1^-} f(x) = 0$

(D) $\lim_{x \rightarrow 1^+} f(x) = 0$

Ans. (A,C)

Sol.
$$f(x) = \begin{cases} (1-x) \cos \frac{1}{1-x}, & x < 1 \\ -(1+x) \cos \frac{1}{1-x}, & x > 1 \end{cases}$$

$$\lim_{x \rightarrow 1^+} f(x) = \text{d.n.e.}, \lim_{x \rightarrow 1^-} f(x) = 0$$

SECTION-3 : (Maximum Marks : 12)

- This section contains **TWO** paragraphs.
- Based on each paragraph, there are **TWO** questions.
- Each question has **FOUR** options (A), (B), (C) and (D) **ONLY ONE** of these four options is correct.
- For each question, darken the bubble corresponding to the correct option in the ORS.
- For each question, marks will be awarded in one of the following categories :

Full Marks : +3 If only the bubble corresponding to the correct option is darkened.

Zero Marks : 0 In all other cases.

PARAGRAPH 1

Let O be the origin, and $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overline{QR}, \overline{RP}, \overline{PQ}$, respectively, of a triangle PQR.

51. $|\overrightarrow{OX} \times \overrightarrow{OY}| =$

(A) $\sin(Q + R)$

(B) $\sin(P + R)$

(C) $\sin 2R$

(D) $\sin(P + Q)$

Ans. (D)

Sol. $\overrightarrow{OX} = \frac{\overrightarrow{QR}}{QR}$

$\overrightarrow{OY} = \frac{\overrightarrow{RP}}{RP}$

$|\overrightarrow{OX} \times \overrightarrow{OY}| = \sin R = \sin(P + Q)$

52. If the triangle PQR varies, then the minimum value of $\cos(P + Q) + \cos(Q + R) + \cos(R + P)$ is

- (A) $\frac{3}{2}$ (B) $-\frac{3}{2}$ (C) $\frac{5}{3}$ (D) $-\frac{5}{3}$

Ans. (B)

Sol. $-(\cos P + \cos Q + \cos R) \geq -\frac{3}{2}$ as we know $\cos P + \cos Q + \cos R$ will take its maximum value when

$P = Q = R = \frac{\pi}{3}$

PARAGRAPH 2

Let p, q be integers and let α, β be the roots of the equation, $x^2 - x - 1 = 0$, where $\alpha \neq \beta$. For $n = 0, 1, 2, \dots$, let $a_n = p\alpha^n + q\beta^n$.

FACT : If a and b are rational numbers and $a + b\sqrt{5} = 0$, then $a = 0 = b$.

53. If $a_4 = 28$, then $p + 2q =$

- (A) 14 (B) 7 (C) 12 (D) 21

Ans. (C)

Sol. $\alpha^2 = \alpha + 1 \Rightarrow \alpha^4 = 3\alpha + 2$

$\therefore a_4 = 28 \Rightarrow p\alpha^4 + q\beta^4 = p(3\alpha + 2) + q(3\beta + 2) = 28$

$\Rightarrow p(3\alpha + 2) + q(3 - 3\alpha + 2) = 28$

$\Rightarrow \alpha(3p - 3q) + 2p + 5q = 28 \quad (\text{as } \alpha \in \mathbb{Q}^c)$

$\Rightarrow p = q, 2p + 5q = 28 \Rightarrow p = q = 4$

$\therefore p + 2q = 12$ **Ans : C**

54. $a_{12} =$

- (A) $2a_{11} + a_{10}$ (B) $a_{11} - a_{10}$ (C) $a_{11} + a_{10}$ (D) $a_{11} + 2a_{10}$

Ans. (C)

Sol. $\alpha^2 = \alpha + 1 \Rightarrow \alpha^n = \alpha^{n-1} + \alpha^{n-2}$

$\Rightarrow p\alpha^n + q\beta^n = p(\alpha^{n-1} + \alpha^{n-2}) + q(\beta^{n-1} + \beta^{n-2})$

$a_n = a_{n-1} + a_{n-2}$

$\Rightarrow a_{12} = a_{11} + a_{10}$