

Ex 12.1

Q1

$P(n) : n(n+1)$ is even

$P(3) : 3.(3+1)$ is even

Q2

$P(n) : n^3 + n$ is divisible by 3

$P(3) : 3^3 + 3$ is divisible by 3

$\Rightarrow P(3) : 30$ is divisible by 3.

$\therefore P(3)$ is true

Now,

$P(4) : 4^3 + 3 = 67$ is divisible by 3

Since, 67 is not divisible by 3

So, $P(4)$ is not true

Q3

$$P(n) : 2^n \geq 3n$$

Given that $P(r)$ is true

$$\Rightarrow 2^r \geq 3r$$

Multiplying both the sides by 2,

$$2 \cdot 2^r \geq 2 \cdot 3r$$

$$2^{r+1} \geq 6r$$

$$\geq 3r + 3r$$

$$\geq 3 + 3r,$$

$$[\text{Since } 3r \geq 3 \Rightarrow 3r + 3r \geq 3 + 3r]$$

$$2^{r+1} \geq 3(r+1)$$

$$\Rightarrow P(r+1) \text{ is true}$$

Q4

Here, $P(n) : n^2 + n$ is even

Given, $P(r)$ is true

$$\Rightarrow r^2 + r \text{ is even}$$

$$\Rightarrow r^2 + r = 2\lambda \quad \text{--- (1)}$$

Now,

$$(r+1)^2 + (r+1)$$

$$= r^2 + 2r + 1 + r + 1$$

$$= (r^2 + r) + 2r + 2$$

$$= 2\lambda + 2r + 2 \quad \text{[Using equation (1)]}$$

$$= 2(\lambda + r + 1)$$

$$= 2\mu$$

$$\Rightarrow (r+1)^2 + (r+1) \text{ is even}$$

$$\Rightarrow P(r+1) \text{ is true}$$

Q5

$$P(n) : 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2} \text{ is true for all } n \in \mathbb{N}$$

Q6

$P(n) : n^2 - n + 41$ is prime

$P(1) : 1 - 1 + 41$ is prime

$\Rightarrow P(1) : 41$ is prime

$\therefore P(1)$ is true.

$P(2) : 2^2 - 2 + 41$ is prime

$\Rightarrow P(2) : 43$ is prime

$\therefore P(2)$ is true.

$P(3) : 3^2 - 3 + 41$ is prime

$\Rightarrow P(3) : 47$ is prime

$\therefore P(3)$ is true.

$P(41) : (41)^2 - 41 + 41$ is prime

$P(41) : (41)^2$ is prime

$\Rightarrow P(41)$ is not true

Ex 12.2

Q1

$$\text{Let } P(n) : 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

For $n = 1$,

$$\text{LHS of } P(n) = 1$$

$$\text{RHS of } P(n) = \frac{1(1+1)}{2} = 1$$

Since, LHS = RHS

$$\Rightarrow P(n) \text{ is true for } n = 1$$

Let $P(n)$ be true for $n = k$, so

$$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2} \quad \dots (1)$$

Now

$$(1 + 2 + 3 + \dots + k) + (k + 1)$$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= (k+1) \left(\frac{k}{2} + 1 \right)$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)[(k+1)+1]}{2}$$

$$\Rightarrow P(n) \text{ is true for } n = k + 1$$

$$\Rightarrow P(n) \text{ is true for all } n \in \mathbb{N}$$

So, by the principle of mathematical induction

$$P(n) : 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2} \text{ is true for all } n \in \mathbb{N}$$

Q2

$$\text{Let } P(n) : 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

For $n = 1$

$$P(1) : 1 = \frac{1(1+1)(2+1)}{6}$$
$$1 = 1$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$P(k) : 1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6} \quad \text{--- (1)}$$

We have to show that $P(n)$ is true for $n = k + 1$

$$\Rightarrow 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\text{So, } 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{[Using equation (1)]}$$

$$= (k+1) \left[\frac{2k^2 + k}{6} + \frac{(k+1)}{1} \right]$$

$$= (k+1) \left[\frac{2k^2 + k + 6k + 6}{6} \right]$$

$$= (k+1) \left[\frac{2k^2 + 7k + 6}{6} \right]$$

$$= (k+1) \left[\frac{2k^2 + 4k + 3k + 6}{6} \right]$$

$$= (k+1) \left[\frac{2k(k+2) + 3(k+2)}{6} \right]$$

$$= \frac{(k+1)(2k+3)(k+2)}{6}$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in \mathbb{N}$ by PMI

Q3

$$\text{Let } P(n) : 1 + 3 + 3^2 + \dots + 3^{n-1} = \frac{3^n - 1}{2}$$

For $n = 1$

$$P(1) : 1 = \frac{3^1 - 1}{2}$$
$$1 = 1$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$

$$1 + 3 + 3^2 + \dots + 3^{k-1} = \frac{3^k - 1}{2} \quad \text{--- (1)}$$

We have to show $P(n)$ is true for $n = k + 1$

$$\text{i.e. } 1 + 3 + 3^2 + \dots + 3^k = \frac{3^{k+1} - 1}{2}$$

Now,

$$\{1 + 3 + 3^2 + \dots + 3^{k-1}\} + 3^{k+1-1}$$

$$= \frac{3^k - 1}{2} + 3^k \quad \text{[Using equation (1)]}$$

$$= \frac{3^k - 1 + 2 \cdot 3^k}{2}$$

$$= \frac{3 \cdot 3^k - 1}{2}$$

$$= \frac{3^{k+1} - 1}{2}$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in N$ by *PMI*

Q4

$$\text{Let } P(n) : \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

For $n = 1$

$$P(1) : \frac{1}{1.2} = \frac{1}{1+1}$$
$$\frac{1}{2} = \frac{1}{2}$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1} \quad \text{--- (1)}$$

We have to show that

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{k(k+1)} + \frac{k}{(k+1)(k+2)} = \frac{k+1}{(k+2)}$$

Now,

$$\left\{ \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{k(k+1)} \right\} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \text{[Using equation (1)]}$$

$$= \frac{1}{k+1} \left[\frac{k(k+2)+1}{(k+2)} \right]$$

$$= \frac{1}{k+1} \left[\frac{k^2+2k+1}{(k+2)} \right]$$

$$= \frac{1}{k+1} \left[\frac{(k+1)(k+1)}{(k+2)} \right]$$

$$= \frac{(k+1)}{(k+2)}$$

$\Rightarrow P(n)$ is true for $n = k+1$

$\Rightarrow P(n)$ is true for all $n \in N$ by PMI

Q5

Let $P(n) : 1 + 3 + 5 + \dots + (2n - 1) = n^2$

For $n = 1$

$$P(1) : 1 = 1^2 \\ 1 = 1$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$P(k) : 1 + 3 + 5 + \dots + (2k - 1) = k^2 \quad \text{--- (1)}$$

We have to show that

$$1 + 3 + 5 + \dots + (2k - 1) + 2(k + 1) - 1 = (k + 1)^2$$

Now,

$$\{1 + 3 + 5 + \dots + (2k - 1)\} + (2k + 1)$$

$$= k^2 + (2k + 1) \quad \text{[Using equation (1)]}$$

$$= k^2 + 2k + 1$$

$$= (k + 1)^2$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in N$ by *PMI*

Q6

$$\text{Let } P(n) : \frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$$

Put $n = 1$

$$P(1) : \frac{1}{2.5} = \frac{1}{6+4}$$
$$\frac{1}{10} = \frac{1}{10}$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{6k+4} \quad \text{--- (1)}$$

We have to show that,

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{(3k+2)(3k+5)} = \frac{(k+1)}{6k+10}$$

Now,

$$\left\{ \frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \frac{1}{(3k-1)(3k+2)} \right\} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{k(3k+5)+2}{2(3k+2)(3k+5)}$$

$$= \frac{3k^2+5k+2}{2(3k+2)(3k+5)}$$

$$= \frac{3k^2+3k+2k+2}{2(3k+2)(3k+5)}$$

$$= \frac{3k(k+1)+2(k+1)}{2(3k+2)(3k+5)}$$

$$= \frac{(k+1)\cancel{(3k+2)}}{2\cancel{(3k+2)}(3k+5)}$$

$$= \frac{(k+1)}{2(3k+5)}$$

$P(n)$ is true for $n = k+1$

$P(n)$ is true for all $n \in \mathbb{N}$ by PMI

Q7

$$\text{Let } P(n) : \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$$

Put $n = 1$

$$P(1) : \frac{1}{1.4} = \frac{1}{4}$$
$$\frac{1}{4} = \frac{1}{4}$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1} \quad \dots (1)$$

We have to show that,

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)} = \frac{(k+1)}{(3k+4)}$$

Now,

$$\left\{ \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} \right\} + \frac{1}{(3k+1)(3k+4)}$$

Now,

$$\left\{ \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} \right\} + \frac{1}{(3k+1)(3k+4)}$$

$$= \frac{k}{(3k+1)} + \frac{1}{(3k+1)(3k+4)}$$

$$= \frac{1}{(3k+1)} \left[\frac{k}{1} + \frac{1}{(3k+4)} \right]$$

$$= \frac{1}{(3k+1)} \left[\frac{k(3k+4) + 1}{(3k+4)} \right]$$

$$= \frac{1}{(3k+1)} \left[\frac{3k^2 + 4k + 1}{(3k+4)} \right]$$

$$= \frac{1}{(3k+1)} \frac{(3k^2 + 3k + k + 1)}{(3k+4)}$$

$$= \frac{3k(k+1) + (k+1)}{(3k+1)(3k+4)}$$

$$= \frac{(k+1)(3k+1)}{(3k+1)(3k+4)}$$

$$= \frac{(k+1)}{(3k+4)}$$

$\Rightarrow P(n)$ is true for $n = k+1$

$\Rightarrow P(n)$ is true for all $n \in N$ by PMI

Q8

$$\text{Let } P(n): \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$$

Put $n = 1$

$$\begin{aligned}\frac{1}{3.5} &= \frac{1}{3(5)} \\ \frac{1}{15} &= \frac{1}{15}\end{aligned}$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2k+1)(2k+3)} = \frac{k}{3(2k+3)} \quad \text{--- (1)}$$

We have to show that,

$$\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2k+1)(2k+3)} + \frac{1}{(2k+3)(2k+5)} = \frac{(k+1)}{3(2k+5)}$$

Now,

$$\left\{ \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2k+1)(2k+3)} \right\} + \frac{1}{(2k+3)(2k+5)}$$

$$= \frac{k}{3(2k+3)} + \frac{1}{(2k+3)(2k+5)} \quad \text{[Using equation (1)]}$$

$$= \frac{1}{(2k+3)} \left[\frac{k}{3} + \frac{1}{(2k+5)} \right]$$

$$= \frac{1}{(2k+3)} \left[\frac{k(2k+5)+3}{(2k+5)} \right]$$

$$= \frac{1}{(2k+3)} \left[\frac{2k^2+5k+3}{(2k+5)} \right]$$

$$= \frac{1}{(2k+3)} \left[\frac{2k^2+2k+3k+3}{(2k+5)} \right]$$

$$= \frac{1}{(2k+3)} \left[\frac{2k(k+1)+3(k+1)}{(2k+5)} \right]$$

$$= \frac{1}{(2k+3)} \left[\frac{(k+1)(2k+3)}{(2k+5)} \right]$$

$$= \frac{(k+1)}{2k+5}$$

$\Rightarrow P(n)$ is true for $n = k+1$

$\Rightarrow P(n)$ is true for all $n \in N$ by *PMI*

Q9

$$\text{Let } P(n) : \frac{1}{3 \cdot 7} + \frac{1}{7 \cdot 11} + \frac{1}{11 \cdot 15} + \dots + \frac{1}{(4n-1)(4n+3)} = \frac{n}{3(4n+3)}$$

For $n = 1$

$$\frac{1}{3 \cdot 7} = \frac{1}{3(7)}$$

$$\frac{1}{21} = \frac{1}{21}$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$\frac{1}{3 \cdot 7} + \frac{1}{7 \cdot 11} + \frac{1}{11 \cdot 15} + \dots + \frac{1}{(4n-1)(4n+3)} = \frac{k}{3(4k+3)} \quad \dots (1)$$

We have to show that,

$$\frac{1}{3 \cdot 7} + \frac{1}{7 \cdot 11} + \frac{1}{11 \cdot 15} + \dots + \frac{1}{(4k-1)(4k+3)} + \frac{1}{(4k+3)(4k+7)} = \frac{(k+1)}{3(4k+7)}$$

Now,

$$\left\{ \frac{1}{3 \cdot 7} + \frac{1}{7 \cdot 11} + \frac{1}{11 \cdot 15} + \dots + \frac{1}{(4k-1)(4k+3)} \right\} + \frac{1}{(4k+3)(4k+7)}$$

Now,

$$\left\{ \frac{1}{3 \cdot 7} + \frac{1}{7 \cdot 11} + \frac{1}{11 \cdot 15} + \dots + \frac{1}{(4k-1)(4k+3)} \right\} + \frac{1}{(4k+3)(4k+7)}$$

$$= \frac{k}{3(4k+3)} + \frac{1}{(4k+3)(4k+7)}$$

$$= \frac{1}{(4k+3)} \left[\frac{k}{3} + \frac{1}{4k+7} \right]$$

$$= \frac{1}{(4k+3)} \left[\frac{k(4k+7)+3}{3(4k+7)} \right]$$

$$= \frac{1}{(4k+3)} \left[\frac{4k^2+7k+3}{3(4k+7)} \right]$$

$$= \frac{1}{(4k+3)} \left[\frac{4k^2+4k+3k+3}{3(4k+7)} \right]$$

$$= \frac{1}{(4k+3)} \left[\frac{4k(k+1)+3(k+1)}{3(4k+7)} \right]$$

$$= \frac{1}{(4k+3)} \left[\frac{(4k+3)(k+1)}{3(4k+7)} \right]$$

$$= \frac{(k+1)}{3(4k+7)}$$

$\Rightarrow P(n)$ is true for $n = k+1$

$\Rightarrow P(n)$ is true for all $n \in \mathbb{N}$ by *PMI*

Q10

Let $P(n) : 1.2 + 2.2^2 + 3.2^3 + \dots + n.2^n = (n-1)2^{n+1} + 2$

For $n = 1$

$$1.2 = 0.2^0 + 2$$

$$2 = 2$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$1.2 + 2.2^2 + 3.2^3 + \dots + k.2^k = (k-1)2^{k+1} + 2 \quad \text{--- (1)}$$

We have to show that,

$$\{1.2 + 2.2^2 + 3.2^3 + \dots + k.2^k\} + (k+1)2^{k+1} = k2^{k+2} + 2$$

Now,

$$\{1.2 + 2.2^2 + 3.2^3 + \dots + k.2^k\} + (k+1)2^{k+1}$$

$$= [(k-1)2^{k+1} + 2] + (k+1)2^{k+1} \quad \text{[Using equation (1)]}$$

$$= (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$$

$$= 2^{k+1}(k-1+k+1) + 2$$

$$= 2^{k+1}.2k + 2$$

$$= k2^{k+2} + 2$$

$\Rightarrow P(n)$ is true for $n = k+1$

$\Rightarrow P(n)$ is true for all $n \in N$ by *PMI*

Q11

$$\text{Let } P(n) : 2 + 5 + 8 + 11 + \dots + (3n - 1) = \frac{1}{2}n(3n + 1)$$

For $n = 1$

$$P(1) \quad 2 = \frac{1}{2} \cdot 1 \cdot (4) \\ 2 = 2$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$2 + 5 + 8 + 11 + \dots + (3k - 1) = \frac{1}{2}k(3k + 1) \quad \text{--- (1)}$$

We have to show that,

$$2 + 5 + 8 + 11 + \dots + (3k - 1) + (3k + 2) = \frac{1}{2}(k + 1)(3k + 4)$$

Now,

$$\{2 + 5 + 8 + 11 + \dots + (3k - 1)\} + (3k + 2)$$

$$= \frac{1}{2}k(3k + 1) + (3k + 2)$$

$$= \frac{3k^2 + k + 2(3k + 2)}{2}$$

$$= \frac{3k^2 + k + 6k + 4}{2}$$

$$= \frac{3k^2 + 7k + 4}{2}$$

$$= \frac{3k^2 + 3k + 4k + k}{2}$$

$$= \frac{3k(k + 1) + 4(k + 1)}{2}$$

$$= \frac{(k + 1)(3k + 4)}{2}$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in \mathbb{N}$ by *PMI*

Q12

$$\text{Let } 1.3 + 2.4 + 3.5 + \dots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$$

For $n = 1$

$$\begin{aligned} 1.3 &= \frac{1}{6} \cdot 1 \cdot (2)(9) \\ 3 &= 3 \end{aligned}$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$1.3 + 2.4 + 3.5 + \dots + k(k+2) = \frac{1}{6}k(k+1)(2k+7) \quad \text{--- (1)}$$

We have to show that,

$$1.3 + 2.4 + 3.5 + \dots + k(k+2) + (k+1)(k+3) = \frac{(k+1)}{6}(k+2)(2k+9)$$

Now,

$$\{1.3 + 2.4 + 3.5 + \dots + k(k+2)\} + (k+1)(k+3)$$

$$= \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3) \quad \text{[Using equation (1)]}$$

$$= (k+1) \left[\frac{k(2k+7)}{6} + \frac{k+3}{1} \right]$$

$$= (k+1) \left[\frac{2k^2 + 7k + 6k + 18}{6} \right]$$

$$= (k+1) \left(\frac{2k^2 + 13k + 18}{6} \right)$$

$$= (k+1) \left[\frac{2k^2 + 4k + 9k + 18}{6} \right]$$

$$= (k+1) \left[\frac{2k(k+2) + 9(k+2)}{6} \right]$$

$$= (k+1) \left[\frac{(2k+9)(k+2)}{6} \right]$$

$$= \frac{1}{6}(k+1)(k+2)(2k+9)$$

$\Rightarrow P(n)$ is true for $n = k+1$

$\Rightarrow P(n)$ is true for all $n \in N$ by PMI

Q13

$$\text{Let } P(n) : 1.3 + 3.5 + 5.7 + \dots + (2n-1)(2n+1) = \frac{n(4n^2 + 6n - 1)}{3}$$

For $n = 1$

$$\begin{aligned} 1.3 &= \frac{1(4 + 6 - 1)}{3} \\ 3 &= 3 \end{aligned}$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$1.3 + 3.5 + 5.7 + \dots + (2k-1)(2k+1) = \frac{k(4k^2 + 6k - 1)}{3} \quad \text{--- (1)}$$

We have to show that,

$$1.3 + 3.5 + 5.7 + \dots + (2k-1)(2k+1) + (2k+1)(2k+3) = \frac{(k+1)[4(k+1)^2 + 6(k+1) - 1]}{3}$$

Now,

$$\{1.3 + 3.5 + 5.7 + \dots + (2k-1)(2k+1)\} + (2k+1)(2k+3)$$

$$= \frac{k(4k^2 + 6k - 1)}{3} + (2k+1)(2k+3) \quad \text{[Using equation (1)]}$$

$$= \frac{k(4k^2 + 6k - 1) + 3(4k^2 + 6k + 2k + 3)}{3}$$

$$= \frac{4k^3 + 6k^2 - k + 12k^2 + 18k + 6k + 9}{3}$$

$$= \frac{4k^3 + 18k^2 + 23k + 9}{3}$$

$$= \frac{4k^3 + 4k^2 + 14k^2 + 14k + 9k + 9}{3}$$

$$= \frac{(k+1)(4k^2 + 8k + 4 + 6k + 6 - 1)}{3}$$

$$= \frac{(k+1)[4(k+1)^2 + 6(k+1) - 1]}{3}$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in N$ by PMI

Q14

$$\text{Let } P(n) : 1.2 + 2.3 + 3.4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

For $n = 1$

$$1.2 = \frac{1(1+1)(1+2)}{3}$$

$$2 = 2$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$

$$\Rightarrow 1.2 + 2.3 + 3.4 + \dots + k(k+1) = \frac{k(k+1)(k+2)}{3} \quad \text{--- (1)}$$

We have to show that,

$$1.2 + 2.3 + 3.4 + \dots + k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

Now,

$$\{1.2 + 2.3 + 3.4 + \dots + k(k+1)\} + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + \frac{(k+1)(k+2)}{1}$$

$$= (k+1)(k+2) \left[\frac{k}{3} + 1 \right]$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in N$ by *PMI*

Q15

$$\text{Let } P(n) : \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

For $n = 1$

$$\frac{1}{2} = 1 - \frac{1}{2^1}$$

$$\frac{1}{2} = \frac{1}{2}$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^k} = 1 - \frac{1}{2^k} \quad \text{--- (1)}$$

We have to show that,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}$$

Now,

$$\left\{ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^k} \right\} + \frac{1}{2^{k+1}}$$

$$= 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \quad \text{[Using equation (1)]}$$

$$= 1 - \left(\frac{2-1}{2^{k+1}} \right)$$

$$= 1 - \frac{1}{2^{k+1}}$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in N$ by *PMI*

Q16

$$\text{Let } P(n) : 1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{1}{3}n(4n^2-1)$$

For $n = 1$

$$\begin{aligned} 1 &= \frac{1}{3} \cdot 1 \cdot (4-1) \\ 1 &= 1 \end{aligned}$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 = \frac{1}{3}k(4k^2-1) \quad \text{--- (1)}$$

We have to show that,

$$1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 + (2k+1)^2 = \frac{1}{3}(k+1)[4(k+1)^2-1]$$

Now,

$$\{1^2 + 3^2 + 5^2 + \dots + (2k-1)^2\} + (2k+1)^2$$

$$= \frac{1}{3}k(4k^2-1) + (2k+1)^2 \quad \text{[Using equation (1)]}$$

$$= \frac{1}{3}k(2k+1)(2k-1) + (2k+1)^2$$

$$= (2k+1) \left[\frac{k(2k-1)}{3} + (2k+1) \right]$$

$$= (2k+1) \left[\frac{2k^2 - k + 3(2k+1)}{3} \right]$$

$$= (2k+1) \left[\frac{2k^2 - k + 6k + 3}{3} \right]$$

$$= \frac{(2k+1)(2k^2+5k+3)}{3}$$

$$= \frac{(2k+1)(2k^2+5k+3)}{3}$$

$$= \frac{(2k+1)(2k(k+1)+3(k+1))}{3}$$

$$= \frac{(2k+1)(2k+3)(k+1)}{3}$$

$$= \frac{(k+1)}{2} [4k^2 + 6k + 2k + 3]$$

$$= \frac{(k+1)}{2} [4k^2 + 8k + 4 - 1]$$

$$= \frac{(k+1)}{2} [4(k+1)^2 - 1]$$

$\Rightarrow P(n)$ is true for $n = k+1$

$\Rightarrow P(n)$ is true for all $n \in N$ by *PMI*

Q17

$$\text{Let } P(n) : a + ar + ar^2 + \dots + ar^{n-1} = a \left(\frac{r^n - 1}{r - 1} \right), r \neq 1$$

For $n = 1$

$$a = a \left(\frac{r^1 - 1}{r - 1} \right)$$

$$a = a$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$a + ar + ar^2 + \dots + ar^{k-1} = a \left(\frac{r^k - 1}{r - 1} \right), r \neq 1 \quad \text{--- (1)}$$

We have to show that,

$$a + ar + ar^2 + \dots + ar^{k-1} + ar^k = a \left(\frac{r^{k+1} - 1}{r - 1} \right)$$

Now,

$$\{a + ar + ar^2 + \dots + ar^{k-1}\} + ar^k$$

$$= a \left(\frac{r^k - 1}{r - 1} \right) + ar^k \quad \text{[Using equation (1)]}$$

$$= \frac{a[r^k - 1 + r^k(r - 1)]}{r - 1}$$

$$= \frac{a[r^k - 1 + r^{k+1} - r^k]}{r - 1}$$

$$= \frac{a(r^{k+1} - 1)}{r - 1}$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in N$ by *PMI*

Q18

$$\text{Let } P(n) : a + (a+d) + (a+2d) + \dots + (a+(n-1)d) = \frac{n}{2}[2a+(n-1)d]$$

For $n = 1$

$$a = \frac{1}{2}[2a + (1-1)d]$$

$$a = a$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$a + (a+d) + (a+2d) + \dots + (a+(k-1)d) = \frac{k}{2}[2a+(k-1)d] \quad \text{--- (1)}$$

We have to show that,

$$a + (a+d) + (a+2d) + \dots + (a+(k-1)d) + (a+kd) = \frac{(k+1)}{2}[2a+kd]$$

Now,

$$\{a + (a+d) + (a+2d) + \dots + (a+(k-1)d)\} + (a+kd)$$

$$= \frac{k}{2}[2a+(k-1)d] + (a+kd) \quad \text{[Using equation (1)]}$$

$$= \frac{2ka + k(k-1)d + 2(a+kd)}{2}$$

$$= \frac{2ka + k^2d - kd + 2a + 2kd}{2}$$

$$= \frac{2ka + 2a + k^2d + kd}{2}$$

$$= \frac{2a(k+1) + d(k^2+k)}{2}$$

$$= \frac{(k+1)}{2}[2a+kd]$$

$\Rightarrow P(n)$ is true for $n = k+1$

$\Rightarrow P(n)$ is true for all $n \in N$ by PMI

Q19

Let $P(n) : (5^{2n} - 1)$ is divisible by 24

For $n = 1$

$$5^2 - 1 = 24$$

Which is divisible by 24

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$

$\Rightarrow (5^{2k} - 1)$ is divisible by 24

$$\Rightarrow 5^{2k} - 1 = 24\lambda \quad \text{--- (1)}$$

We have to show that,

$(5^{2k} - 1)$ is divisible by 24

$$5^{2(k+1)} - 1 = 24\mu$$

Now,

$$5^{2(k+1)} - 1$$

$$= 5^{2k} \cdot 5^2 - 1$$

$$= 25 \cdot 5^{2k} - 1$$

$$= 25(24\lambda + 1) - 1 \quad \text{[Using equation (1)]}$$

$$= 25 \cdot 24\lambda + 24$$

$$= 24(25\lambda + 1)$$

$$= 24\mu$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in N$ by PMI

Q20

Let $P(n) : 3^{2n} + 7$ is divisible by 8

For $n = 1$

$$3^2 + 7 = 16$$

Which is divisible by 8

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$3^{2k} + 7$ is divisible by 8

$$\Rightarrow 3^{2k} + 7 = 8\lambda \quad \text{--- (1)}$$

We have to show that,

$3^{2(k+1)} + 7$ is divisible by 8

$$3^{2(k+1)} + 7 = 8\mu$$

Now,

$$3^{2(k+1)} + 7$$

$$= 3^{2k} \cdot 3^2 + 7$$

$$= 9 \cdot 3^{2k} + 7$$

$$= 9 \cdot (8\lambda - 7) + 7$$

$$= 72\lambda - 56$$

$$= 8(9\lambda - 7)$$

$$= 8\mu$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in N$ by *PMI*

Q21

Let $P(n) : 5^{2n+2} - 24n - 25$ is divisible by 576

For $n = 1$

$$5^4 - 24 - 25$$

$$= 625 - 49$$

$$= 576$$

Which is divisible by 576

Let $P(n)$ is true for $n = k$, so

$5^{2k+2} - 24k - 25$ is divisible by 576

$$5^{2k+2} - 24k - 25 = 576\lambda \quad \text{--- (1)}$$

We have to show that,

$5^{2k+4} - 24(k+1) - 25$ is divisible by 576

$$5^{(2k+2)+2} - 24(k+1) - 25 = 576\mu$$

Now,

$$5^{(2k+2)+2} - 24(k+1) - 25$$

$$= 5^{(2k+2)} \cdot 5^2 - 24k - 24 - 25$$

$$= (576\lambda + 24k + 25) 25 - 24k - 49 \quad \text{[Using equation (1)]}$$

$$= 25 \cdot 576\lambda + 600k + 625 - 24k - 49$$

$$= 25 \cdot 576\lambda + 576k + 576$$

$$= 576(25\lambda + k + 1)$$

$$= 576\mu$$

$$\Rightarrow P(n) \text{ is true for } n = k + 1$$

$$\Rightarrow P(n) \text{ is true for all } n \in N \text{ by PMI}$$

Q22

Let $P(n) : 3^{2n+2} - 8n - 9$ is divisible by 8

For $n = 1$

$$3^{2+2} - 8 - 9$$

$$= 81 - 17$$

$$= 64$$

It is divisible by 8

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$(3^{2k+2} - 8k - 9)$ is divisible by 8

$$\Rightarrow 3^{2k+2} - 8k - 9 = 8\lambda \quad \text{--- (1)}$$

We have to show that,

$3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 8

$$3^{2(k+1)+2} - 8(k+1) - 9 = 8\mu$$

Now,

$$3^{2(k+1)+2} - 8(k+1) - 9$$

$$= (8\lambda + 8k + 9) - 8(k+1) - 9$$

$$= 8\lambda + 8k + 9 - 8k - 8 - 9$$

$$= 8\lambda + 64k + 64$$

$$= 8(8\lambda + 8k + 8)$$

$$= 8\mu$$

$\Rightarrow P(n)$ is true for $n = 1$

$\Rightarrow P(n)$ is true for all $n \in \mathbb{N}$ by PMI

Q23

Let $P(n) : (ab)^n = a^n b^n$

For $n = 1$

$$(ab)^1 = a^1 b^1$$

$$ab = ab$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$,

$$(ab)^k = a^k b^k \quad \text{--- (1)}$$

We have to show that,

$$(ab)^{k+1} = a^{k+1} b^{k+1}$$

Now,

$$(ab)^{k+1}$$

$$= (ab)^k (ab)$$

$$= (a^k b^k) (ab) \quad \text{[Using equation (1)]}$$

$$= (a^{k+1}) (b^{k+1})$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in \mathbb{N}$ by *PMI*

Q24

Let $P(n) : n(n+1)(n+5)$ is a multiple of 3 for all $n \in \mathbb{N}$

For $n = 1$

$$\begin{aligned} & 1.(1+1)(1+5) \\ &= (2)(6) \end{aligned}$$

$$= 12$$

it is a multiple of 3

Let $P(n)$ is true for $n = k$

$k(k+1)(k+5)$ is a multiple of 3

$$k(k+1)(k+5) = 3\lambda \quad \text{--- (1)}$$

We have to show that,

$(k+1)[(k+1)+1][(k+1)+5]$ is a multiple of 3

$$(k+1)[(k+1)+1][(k+1)+5] = 3\mu$$

Now,

$$\begin{aligned} & (k+1)(k+2)[(k+1)+5] \\ &= [k(k+1)+2(k+1)][(k+5)+1] \\ &= k(k+1)(k+5)+k(k+1)+2(k+1)(k+5)+2(k+1) \\ &= 3\lambda+k^2+k+2(k^2+6k+5)+2k+2 \quad \text{[Using equation (1)]} \\ &= 3\lambda+k^2+k+2k^2+12k+10+2k+2 \\ &= 3\lambda+3k^2+15k+12 \\ &= 3(\lambda+k^2+5k+4) \\ &= 3\mu \end{aligned}$$

$$\Rightarrow P(n) \text{ is true for } n = k+1$$

$$\Rightarrow P(n) \text{ is true for all } n \in \mathbb{N} \text{ by PMI}$$

Q25

Let $P(n) : 7^{2n} + 2^{3n-3} \cdot 3^{n-1}$ is divisible by 25

For $n = 1$

$$7^2 + 2^0 \cdot 3^0$$

$$= 49 + 1$$

$$= 50$$

it is divisible of 25

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$,

$7^{2k} + 2^{3k-3} \cdot 3^{k-1}$ is divisible by 25

$$\Rightarrow 7^{2k} + 2^{3k-3} \cdot 3^{k-1} = 25\lambda \quad \text{--- (1)}$$

We have to show that,

$7^{2(k+1)} + 2^{3k} \cdot 3^k$ is divisible by 25

$$7^{2(k+1)} + 2^{3k} \cdot 3^k = 25\mu$$

Now,

$$7^{2(k+1)} + 2^{3k} \cdot 3^k$$

$$= 7^{2k} \cdot 7^2 + 2^{3k} \cdot 3^k$$

$$= (25\lambda - 2^{3k-3} \cdot 3^{k-1}) 49 + 2^{3k} \cdot 3^k \quad \text{[Using equation (1)]}$$

$$= 25\lambda \cdot 49 - \frac{2^{3k}}{8} \cdot \frac{3^k}{3} \cdot 49 + 2^{3k} \cdot 3^k$$

$$= 24 \cdot 25 \cdot 49\lambda - 2^{3k} \cdot 3^k \cdot 49 + 24 \cdot 2^{3k} \cdot 3^k$$

$$= 24 \cdot 25 \cdot 49\lambda - 25 \cdot 2^{3k} \cdot 3^k$$

$$= 25 (24 \cdot 49\lambda - 2^{3k} \cdot 3^k)$$

$$= 25\mu$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in \mathbb{N}$ by PMI

Q26

Let $P(n) : 2 \cdot 7^n + 3 \cdot 5^n - 5$ is divisible by 24

For $n = 1$

$$2 \cdot 7 + 3 \cdot 5 - 5$$

$$= 24$$

it is divisible of 24

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$2 \cdot 7^k + 3 \cdot 5^k - 5$ is divisible by 24

$$2 \cdot 7^k + 3 \cdot 5^k - 5 = 24\lambda \quad \text{--- (1)}$$

We have to show that,

$$2 \cdot 7^{(k+1)} + 3 \cdot 5^{(k+1)} - 5$$

$$= 2 \cdot 7^k \cdot 7 + 3 \cdot 5^k \cdot 5 - 5$$

$$= (24\lambda - 3 \cdot 5^k + 5)7 + 15 \cdot 5^k - 5$$

$$= 24 \cdot 7\lambda - 21 \cdot 5^k + 35 + 15 \cdot 5^k - 5$$

$$= 24 \cdot 7\lambda - 6 \cdot 5^k + 30$$

$$= 24 \cdot 7\lambda - 6(5^k - 5)$$

$$= 24 \cdot 7\lambda - 6 \cdot (20\nu) \quad \left[\text{Since } 5^k - 5 \text{ is multiple of } 20 \right]$$

$$= 24(7\lambda - 5\nu)$$

$$= 24\mu$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in \mathbb{N}$ by PMI

Q27

Let $P(n) : 11^{n+2} + 12^{2n+1}$ is divisible by 133

For $n = 1$

$$\begin{aligned} 11^3 + 12^3 \\ = 1331 + 1728 \end{aligned}$$

$$= 3059$$

it is divisible of 133

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$11^{k+2} + 12^{2k+1}$ is divisible by 133

$$11^{k+2} + 12^{2k+1} = 133\lambda \quad \text{--- (1)}$$

We have to show that,

$11^{k+3} + 12^{2k+3}$ is divisible by 133

Now,

$$\begin{aligned} 11^{k+2} \cdot 11 + 12^{2k+1} \cdot 12^2 \\ = (133\lambda - 12^{2k+1}) \cdot 11 + 12^{2k+1} \cdot 144 \\ = 11 \cdot 133\lambda - 11 \cdot 12^{2k+1} + 144 \cdot 12^{2k+1} \\ = 11 \cdot 133\lambda + 133 \cdot 12^{2k+1} \\ = 133 (11\lambda + 12^{2k+1}) \end{aligned}$$

$$= 133\mu$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in N$ by *PMI*

Q28

Consider equation

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n!$$

Lets take $(n+1)! - n! = n! (n+1 - 1) = n \times n!$

Now substitue $n=1,2,3,4,\dots,n$ in above equation we get

$$2! - 1! = 1 \times 1!$$

$$3! - 2! = 2 \times 2!$$

$$4! - 3! = 3 \times 3!$$

.....

$$(n+1)! - n! = n \times n!$$

Adding all the above terms gives

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = 2! - 1! + 3! - 2! + 4! - 3! + \dots + (n+1)! - n!$$

$$1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n+1)! - 1$$

Q29

Let $P(n)$ be the statement given by

$P(n) : n^3 - 7n + 3$ is divisible by 3.

Step I:

$P(1) : 1^3 - 7(1) + 3$ is divisible by 3

$\therefore 1 - 7 + 3 = -3$ is divisible by 3

$\therefore P(1)$ is true.

Step II:

Let $P(m)$ is true. Then,

$m^3 - 7m + 3$ is divisible by 3

$\Rightarrow m^3 - 7m + 3 = 3\lambda$ for some $\lambda \in \mathbb{N} \dots (i)$

We have to prove that $P(m+1)$ is true.

$$\begin{aligned}(m+1)^3 - 7(m+1) + 3 &= m^3 + 3m^2 + 3m + 1 - 7m - 7 + 3 \\&= m^3 - 7m + 3 + 3m^2 + 3m + 1 - 7 \\&= (m^3 - 7m + 3) + 3(m^2 + m - 2) \\&= 3\lambda + 3(m^2 + m - 2) \dots \dots \dots [Using (i)] \\&= 3[\lambda + (m^2 + m - 2)] \text{ which is divisible by 3}\end{aligned}$$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in \mathbb{N}$.

Q30

Let $P(n)$ be the statement given by

$$P(n) : 1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1 \text{ for all } n \in \mathbb{N}.$$

Step I:

$$P(1) : 1 + 2^1 = 2^{1+1} - 1$$

$$\Rightarrow 1 + 2 = 4 - 1$$

$$\Rightarrow 3 = 3$$

$\therefore P(1)$ is true.

Step II:

Let $P(m)$ is true. Then,

$$1 + 2 + 2^2 + \dots + 2^m = 2^{m+1} - 1 \dots\dots (i)$$

We have to prove that $P(m+1)$ is true.

$$\begin{aligned} 1 + 2 + 2^2 + \dots + 2^{m+1} &= 1 + 2 + 2^2 + \dots + 2^m + 2^{m+1} \\ &= (2^{m+1} - 1) + 2^{m+1} \dots\dots\dots [Using (i)] \\ &= (2^{m+1} + 2^{m+1}) - 1 \\ &= 2 \times 2^{m+1} - 1 \\ &= 2^{m+2} - 1 \end{aligned}$$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in \mathbb{N}$.

Q31

Let $P(n)$ be the statement given by

$$P(n) : \underbrace{7 + 77 + 777 + \dots + 777 \dots 7}_{n \text{ - digits}} = \frac{7}{81} [10^{n+1} - 9n - 10] \text{ for all } n \in \mathbb{N}.$$

Step I:

$$P(1) : 7 = \frac{7}{81} [10^{1+1} - 9(1) - 10]$$

$$\Rightarrow 7 = \frac{7}{81} \times (100 - 9 - 10)$$

$$\Rightarrow 7 = \frac{7}{81} \times 81$$

$$\Rightarrow 7 = 7 \times (1)$$

$\therefore P(1)$ is true.

Step II:

Let $P(m)$ is true. Then,

$$\underbrace{7 + 77 + 777 + \dots + 777 \dots 7}_{m \text{ - digits}} = \frac{7}{81} [10^{m+1} - 9m - 10] \dots \dots (i)$$

We have to prove that $P(m+1)$ is true.

$$\underbrace{7 + 77 + 777 + \dots + 777 \dots 7}_{m+1 \text{ - digits}} = \underbrace{7 + 77 + 777 + \dots + 777 \dots 7}_{m \text{ - digits}} + \underbrace{777 \dots 7}_{m+1 \text{ - digits}}$$

$$= \frac{7}{81} [10^{m+1} - 9m - 10] + 7 \underbrace{[1111 \dots 1]}_{m+1 \text{ - digits}} \quad [\text{Using (i)}]$$

$$= \frac{7}{81} [10^{m+1} - 9m - 10] + \frac{7}{9} \underbrace{[9999 \dots 9]}_{m+1 \text{ - digits}}$$

$$= \frac{7}{81} [10^{m+1} - 9m - 10] + \frac{7}{9} [10^{m+1} - 1]$$

$$= \frac{7}{81} [(1+9)10^{m+1} - 9m - 19]$$

$$= \frac{7}{81} [10 \times 10^{m+1} - 9(m+1) - 10]$$

$$= \frac{7}{81} [10^{m+2} - 9(m+1) - 10]$$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in \mathbb{N}$.

Q32

Let $P(n) : \frac{n^7}{7} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{n^2}{2} - \frac{37}{210}$ n is a positive integer

For $n = 1$

$$\begin{aligned} & \frac{1}{7} + \frac{1}{5} + \frac{1}{3} + \frac{1}{2} - \frac{37}{210} \\ &= \frac{30 + 42 + 70 + 105 - 37}{210} \\ &= \frac{247 - 37}{210} \end{aligned}$$

It is a positive integer

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$,

$$\begin{aligned} & \frac{k^7}{7} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{k^2}{2} - \frac{37}{210} k \text{ is positive integer} \\ & \frac{k^7}{7} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{k^2}{2} - \frac{37}{210} k = \lambda \end{aligned}$$

For $n = k + 1$,

$$\begin{aligned} & \frac{(k+1)^7}{7} + \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{(k+1)^2}{2} - \frac{37}{210}(k+1) \\ &= \frac{1}{7}[k^7 + 7k^6 + 21k^5 + 35k^4 + 35k^3 + 21k^2 + 7k + 1] + \frac{1}{5}[k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1] \\ & \quad + \frac{1}{3}[k^3 + 3k^2 + 3k + 1] + \frac{1}{2}[k^2 + 2k + 1] - \frac{37k}{210} - \frac{37}{210} \\ &= \left[\frac{k^7}{7} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{k^2}{2} - \frac{37k}{210} \right] + \left[k^6 + 3k^5 + 5k^4 + 5k^3 + 3k^2 + k + \frac{1}{7} + k^4 + 2k^3 + 2k^2 + \frac{1}{5} + k^2 \right. \\ & \quad \left. + k + \frac{1}{3} + k + \frac{1}{2} - \frac{37}{210} \right] \\ &= \lambda + k^6 + 3k^5 + 6k^4 + 7k^3 + 6k^2 + 3k + \frac{1}{7} + \frac{1}{5} + \frac{1}{3} + \frac{1}{2} - \frac{37}{210} \\ &= \lambda + k^6 + 3k^5 + 6k^4 + 7k^3 + 6k^2 + 3k + 1 \\ &= \text{Positive integer} \end{aligned}$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in N$ by PMI

Q33

$P(n): \frac{n^{11}}{11} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{62}{165}n$ is a positive integer

For $n = 1$

$$\begin{aligned} & \frac{1}{11} + \frac{1}{5} + \frac{1}{3} + \frac{62}{165} \\ &= \frac{15 + 33 + 55 + 62}{165} \\ &= \frac{165}{165} \end{aligned}$$

Which is a positive integer

Let $P(n)$ is true for $n = k$, so

$\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165}$ is a positive integer

$$\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165} = \lambda \quad \text{--- (i)}$$

For $n = k + 1$

$$\begin{aligned} & \frac{(k+1)^{11}}{11} + \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{62}{165}(k+1) \\ &= \frac{1}{11} [k^{11} + 11k^{10} + 55k^9 + 165k^8 + 330k^7 + 462k^6 + 462k^5 + 330k^4 + 165k^3 + 55k^2 + 11k + 1] \\ & \quad + \frac{1}{5} [k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1] + \frac{1}{3} [k^3 + 3k^2 + 3k + 1] + \frac{62}{165} [k + 1] \\ &= \left[\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165} \right] + k^{10} + 5k^9 + 15k^8 + 30k^7 + 42k^6 + 42k^5 + 30k^4 + 15k^3 + 5k^2 + 1 + \frac{1}{11} \\ & \quad + k^4 + 2k^3 + 2k^2 + k + \frac{1}{5} + k^2 + k + \frac{1}{3} + \frac{62}{165} \\ &= \lambda + k^{10} + 5k^9 + 15k^8 + 30k^7 + 42k^6 + 42k^5 + 31k^4 + 17k^3 + 8k^2 + 2k + 1 \\ &= \text{An integer} \end{aligned}$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in N$ by PMI

Q34

$$\text{Let } P(n) : \frac{1}{2} \tan\left(\frac{x}{2}\right) + \frac{1}{4} \tan\left(\frac{x}{4}\right) + \dots + \frac{1}{2^n} \tan\left(\frac{x}{2^n}\right) = \frac{1}{2^n} \cot\left(\frac{x}{2^n}\right) - \cot x$$

For $n = 1$

$$\frac{1}{2} \tan \frac{x}{2} = \frac{1}{2} \cot\left(\frac{x}{2}\right) - \cot x$$

$$\begin{aligned} &= \frac{1}{2} \frac{1}{\tan \frac{x}{2}} - \frac{1}{\tan x} \\ &= \frac{1}{2 \tan \frac{x}{2}} - \frac{1}{\left(\frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} \right)} \\ &= \frac{1}{2 \tan \frac{x}{2}} - \frac{1 - \tan^2 \frac{x}{2}}{2 \tan \frac{x}{2}} \\ &= \frac{1 - 1 + \tan^2 \frac{x}{2}}{2 \tan \frac{x}{2}} \\ &= \frac{\tan^2 \frac{x}{2}}{2 \tan \frac{x}{2}} \\ &= \frac{1}{2} \tan \frac{x}{2} \end{aligned}$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$\frac{1}{2} \tan\left(\frac{x}{2}\right) + \frac{1}{4} \tan\left(\frac{x}{4}\right) + \dots + \frac{1}{2^k} \tan\left(\frac{x}{2^k}\right) = \frac{1}{2^k} \cot\left(\frac{x}{2^k}\right) - \cot x \quad \text{----(1)}$$

We have to show that,

$$\frac{1}{2} \tan \frac{x}{2} + \frac{1}{4} \tan \left(\frac{x}{4} \right) + \dots + \frac{1}{2^k} \tan \left(\frac{x}{2^k} \right) + \frac{1}{2^{k+1}} \tan \left(\frac{x}{2^{k+1}} \right) = \frac{1}{2^{k+1}} \cot \left(\frac{x}{2^{k+1}} \right) - \cot x$$

Now,

$$\begin{aligned} & \left\{ \frac{1}{2} \tan \frac{x}{2} + \frac{1}{4} \tan \left(\frac{x}{4} \right) + \dots + \frac{1}{2^k} \tan \left(\frac{x}{2^k} \right) \right\} + \frac{1}{2^{k+1}} \tan \left(\frac{x}{2^{k+1}} \right) \\ &= \frac{1}{2^k} \cot \left(\frac{x}{2^k} \right) - \cot x + \frac{1}{2^{k+1}} \tan \left(\frac{x}{2^{k+1}} \right) \\ &= \frac{1}{2^k} \cot \left(\frac{x}{2^k} \right) - \cot x + \frac{1}{2 \cdot 2^k} \frac{1}{\cot \left(\frac{x}{2^k} \cdot \frac{1}{2} \right)} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2^k} \left[\frac{1}{\tan \left(\frac{x}{2^k} \right)} + \frac{1}{2} \cdot \tan \left\{ \left(\frac{x}{2^k} \right) \cdot \frac{1}{2} \right\} \right] - \cot x \\ &= \frac{1}{2^k} \left[\frac{1 - \tan^2 \left(\frac{x}{2^{k+1}} \right)}{2 \tan \left(\frac{x}{2^{k+1}} \right)} + \frac{1}{2} \tan \left(\frac{x}{2 \cdot 2^k} \right) \right] - \cot x \\ &= \frac{1}{2^k} \left[\frac{1 - \tan^2 \left(\frac{x}{2^{k+1}} \right) + \tan^2 \left(\frac{x}{2^{k+1}} \right)}{2 \tan \left(\frac{x}{2^{k+1}} \right)} \right] - \cot x \\ &= \frac{1}{2^{k+1}} \left[\frac{1}{\tan \left(\frac{x}{2^{k+1}} \right)} \right] - \cot x \end{aligned}$$

$$= \frac{1}{2^{k+1}} \cot \left(\frac{x}{2^{k+1}} \right) - \cot x$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in N$ by *PMI*

Q35

$$\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \dots \left(1 - \frac{1}{n^2}\right)$$

Above can be written as

$$\begin{aligned} &= \left(\frac{2^2-1}{2^2}\right) \left(\frac{3^2-1}{3^2}\right) \left(\frac{4^2-1}{4^2}\right) \dots \left(\frac{n^2-1}{n^2}\right) \\ &= \left(\frac{(2+1)(2-1)}{2^2}\right) \left(\frac{(3+1)(3-1)}{3^2}\right) \\ &\quad \left(\frac{(4+1)(4-1)}{4^2}\right) \dots \left(\frac{(n+1)(n-1)}{n^2}\right) \\ &= \left(\frac{3 \cdot 1}{2^2}\right) \left(\frac{4 \cdot 2}{3^2}\right) \left(\frac{5 \cdot 3}{4^2}\right) \dots \left(\frac{(n+1) \cdot (n-1)}{n^2}\right) \end{aligned}$$

In the above product, there are two series in numerator

$3 \cdot 4 \cdot 5 \dots (n+1)$ and $1 \cdot 2 \cdot 3 \dots (n-1)$

All numbers from 3 to $(n-1)$ are repeated twice

and 1, 2, n are appeared once in numerator

So after cancelling like terms we get

$$= \frac{(n+1)}{2n}$$

Q36

$$P(n) : \frac{(2n)!}{2^{2n} (n!)^2} \leq \frac{1}{\sqrt{3n+1}}$$

For $n = 1$

$$\frac{2!}{2^2 \cdot 1} \leq \frac{1}{\sqrt{4}}$$

$$= \frac{1}{2} \leq \frac{1}{2}$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$\frac{(2k)!}{2^{2k} (k!)^2} \leq \frac{1}{\sqrt{3k+1}} \quad \text{--- (1)}$$

We have to show that,

$$\frac{2(k+1)!}{2^{2(k+1)} [(k+1)!]^2} \leq \frac{1}{\sqrt{3k+4}}$$

Now,

$$\frac{2(k+1)!}{2^{2(k+1)} [(k+1)!]^2}$$

$$= \frac{(2k+2)!}{2^{2k} \cdot 2^2 (k+1)! (k+1)!}$$

$$= \frac{(2k+2)(2k+1)(2k)!}{4 \cdot 2^2 (k+1)(k!) (k+1)(k!)}$$

$$= \frac{2(k+1)(2k+1)(2k)!}{4 \cdot (k+1)^2 \cdot 2^{2k} \cdot (k!)^2}$$

$$\leq \frac{2(2k+1)}{4(k+1)} \cdot \frac{1}{\sqrt{3k+1}} \quad \text{[Using equation (1)]}$$

$$\leq \frac{(2k+1)}{2(k+1)} \cdot \frac{1}{\sqrt{3k+1}}$$

$$\leq \frac{(2k+2)}{2(k+1)} \cdot \frac{1}{\sqrt{3k+3+1}}$$

$$\leq \frac{1}{\sqrt{3k+4}} \quad \left[\begin{array}{l} \text{Since, } 2k+1 < 2k+2 \\ 3k+1 \leq 3k+4 \end{array} \right]$$

$\Rightarrow P(n)$ is true for $n = k+1$

$\Rightarrow P(n)$ is true for all $n \in \mathbb{N}$ by PMI

Q37

Let $P(n) : 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all $n \geq 2$

For $n = 2$

$$1 + \frac{1}{4} < 2 - \frac{1}{4}$$

$$= \frac{5}{4} < \frac{7}{4}$$

$\Rightarrow P(n)$ is true for $n = 2$

Let $P(n)$ is true for $n = k$,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots + \frac{1}{k^2} < 2 - \frac{1}{k} \quad \text{--- (1)}$$

Now, we have to show that,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{(k+1)}$$

Now,

$$\begin{aligned} & 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \\ & < 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \quad [\text{Using (1)}] \\ & < 2 - \frac{k^2 + 2k + 1 - k}{k(k+1)^2} \\ & < 2 - \frac{k^2 + k + 1}{k(k+1)^2} \\ & < 2 - \frac{k^2 + k}{k(k+1)^2} \\ & < 2 - \frac{k(k+1)}{k(k+1)^2} \\ & < 2 - \frac{1}{k+1} \end{aligned}$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in N$ by PMI

Q38

Let $P(n) : x^{2n-1} + y^{2n-1}$ is divisible by $(x + y)$

For $n = 1$

$$x^{2(1)-1} + y^{2(1)-1}$$

$$= x + y$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$,

$x^{2k-1} + y^{2k-1}$ is divisible by $(x + y)$

$$x^{2k-1} + y^{2k-1} = (x + y) \lambda \quad \text{--- (1)}$$

We have to show that,

$$x^{2k+1} + y^{2k+1} = (x + y) \mu$$

Now,

$$x^{2k+1} + y^{2k+1}$$

$$= x^{2k-1} x^2 + y^{2k-1} y^2$$

$$= \left[(x + y) \lambda - y^{2k-1} \right] x^2 + y^{2k-1} y^2$$

$$= (x + y) \lambda x^2 - y^{2k-1} x^2 + y^{2k-1} y^2$$

$$= (x + y) \lambda x^2 - y^{2k-1} (x^2 - y^2)$$

$$= (x + y) \lambda x^2 - y^{2k-1} (x + y) (x - y)$$

$$= (x + y) \left[\lambda x^2 - y^{2k-1} (x - y) \right]$$

$$= (x + y) \mu$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in N$ by *PMI*

Q39

$$\text{Let } P(n) : \sin x + \sin 3x + \dots + \sin(2n-1)x = \frac{\sin^2 nx}{\sin x}$$

For $n = 1$

$$\sin x = \frac{\sin^2 x}{\sin x}$$

$$\sin x = \sin x$$

$$\Rightarrow P(n) \text{ is true for } n = 1$$

Let $P(n)$ is true for $n = k$, so

$$\sin x + \sin 3x + \dots + \sin(2k-1)x = \frac{\sin^2 kx}{\sin x} \quad \text{--- (i)}$$

We have to show that

$$\sin x + \sin 3x + \dots + \sin(2k-1)x + \sin(2k+1)x = \frac{\sin^2 (k+1)x}{\sin x}$$

Now,

$$\begin{aligned} & \{\sin x + \sin 3x + \dots + \sin(2k-1)x\} + \sin(2k+1)x \\ &= \frac{\sin^2 kx}{\sin x} + \frac{\sin(2k+1)x}{1} \end{aligned}$$

Using equation (i),

$$\begin{aligned} &= \frac{\sin^2 kx + \sin(2k+1)x \sin x}{\sin x} \\ &= \frac{2 \sin^2 kx + \cos[(2k+1)x - x] - \cos[2kx + x + x]}{2 \sin x} \\ &= \frac{2 \sin^2 kx + \cos 2kx - \cos(2kx + 2x)}{2 \sin x} \\ &= \frac{1 - \cos 2kx + \cos 2kx - \cos 2x(k+1)}{2 \sin x} \\ &= \frac{1 - \cos 2x(k+1)}{2 \sin x} \\ &= \frac{2 \sin^2 x(k+1)}{2 \sin x} \\ &= \frac{\sin^2 x(k+1)}{\sin x} \end{aligned}$$

$$\Rightarrow P(n) \text{ is true for } n = k+1$$

$$\Rightarrow P(n) \text{ is true for all } n \in N \text{ by PMI.}$$

Q40

Let $P(n)$ be the statement given by

$$P(n) : \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos(\alpha + (n-1)\beta) \\ = \frac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\} \sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}} \text{ for all } n \in \mathbb{N}.$$

Step I:

$$P(1) : \cos \alpha = \frac{\cos\left\{\alpha + \left(\frac{1-1}{2}\right)\beta\right\} \sin\left(\frac{1\beta}{2}\right)}{\sin\frac{\beta}{2}} \\ \Rightarrow \cos \alpha = \frac{\cos\{\alpha + 0\} \sin\left(\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}} \\ \Rightarrow \cos \alpha = \cos \alpha$$

$\therefore P(1)$ is true.

Step II:

Let $P(m)$ is true. Then,

$$\cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos(\alpha + (m-1)\beta) \\ = \frac{\cos\left\{\alpha + \left(\frac{m-1}{2}\right)\beta\right\} \sin\left(\frac{m\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

We have to prove that $P(m+1)$ is true.

$$\cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos(\alpha + (m)\beta) \\ = \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos(\alpha + (m-1)\beta) + \cos(\alpha + (m)\beta) \\ = \frac{\cos\left\{\alpha + \left(\frac{m-1}{2}\right)\beta\right\} \sin\left(\frac{m\beta}{2}\right)}{\sin\frac{\beta}{2}} + \cos(\alpha + (m)\beta) \dots \dots \dots [\text{Using (i)}]$$

$$\begin{aligned}
&= \frac{\sin \frac{\beta}{2} \cos \left(\alpha + (m)\beta \right) + \cos \left\{ \alpha + \left(\frac{m-1}{2} \right) \beta \right\} \sin \left(\frac{m\beta}{2} \right)}{\sin \frac{\beta}{2}} \\
&= \frac{\frac{1}{2} \left[\sin \left(\alpha + \left(\frac{2m+1}{2} \right) \beta \right) - \sin \left(\alpha + \left(\frac{2m-1}{2} \right) \beta \right) \right] + \cos \left\{ \alpha + \left(\frac{m-1}{2} \right) \beta \right\} \sin \left(\frac{m\beta}{2} \right)}{\sin \frac{\beta}{2}} \\
&= \frac{\frac{1}{2} \left[\sin \left(\alpha + \left(\frac{2m+1}{2} \right) \beta \right) - \sin \left(\alpha + \left(\frac{2m-1}{2} \right) \beta \right) \right] + \frac{1}{2} \left[\sin \left(\alpha + \left(\frac{2m-1}{2} \right) \beta \right) + \sin \left(-\alpha + \frac{\beta}{2} \right) \right]}{\sin \frac{\beta}{2}} \\
&= \frac{\frac{1}{2} \left[\sin \left(\alpha + \left(\frac{2m+1}{2} \right) \beta \right) \right] + \frac{1}{2} \left[\sin \left(-\alpha + \frac{\beta}{2} \right) \right]}{\sin \frac{\beta}{2}} \\
&= \frac{\frac{1}{2} \left[\sin \alpha \cos \left(\frac{2m+1}{2} \right) \beta + \cos \alpha \sin \left(\frac{2m+1}{2} \right) \beta + \sin \frac{\beta}{2} \cos \alpha - \cos \frac{\beta}{2} \sin \alpha \right]}{\sin \frac{\beta}{2}} \\
&= \frac{\frac{1}{2} \left[\sin \alpha \left(\cos \left(\frac{2m+1}{2} \right) \beta - \cos \frac{\beta}{2} \right) + \cos \alpha \left(\sin \left(\frac{2m+1}{2} \right) \beta + \sin \frac{\beta}{2} \right) \right]}{\sin \frac{\beta}{2}} \\
&= \frac{\frac{1}{2} \left[-2 \sin \alpha \left(\left(\sin \left(\frac{m+1}{2} \right) \beta \right) \sin \frac{m\beta}{2} \right) + 2 \cos \alpha \left(\left(\sin \left(\frac{m+1}{2} \right) \beta \right) \cos \frac{m\beta}{2} \right) \right]}{\sin \frac{\beta}{2}} \\
&= \frac{\sin \left(\frac{(m+1)\beta}{2} \right) \left[\cos \alpha \cos \frac{m\beta}{2} - \sin \alpha \sin \left(\frac{(m+1)\beta}{2} \right) \right]}{\sin \frac{\beta}{2}} \\
&= \frac{\sin \left(\frac{(m+1)\beta}{2} \right) \cos \left(\alpha + \frac{m\beta}{2} \right)}{\sin \frac{\beta}{2}}
\end{aligned}$$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in \mathbb{N}$.

Q41

$$\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{13}{24},$$

Using induction we first show this is true for $n=2$:

$$\frac{1}{3} + \frac{1}{4} = \frac{7}{12} = \frac{14}{24} > \frac{13}{24} \text{ (True)}$$

Now let's assume it is true for some $n=k$,

$$S_k = \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k} > \frac{13}{24}$$

Finally we need to prove that this implies

it is also true for $n=k+1$:

$$\begin{aligned} S_{k+1} &= \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k+2} \\ &= \frac{-1}{k+1} + \frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2} \\ &= \frac{-1}{k+1} + S_k + \frac{1}{2k+1} + \frac{1}{2k+2} \\ &= S_k + \frac{1}{2(2k+1)(k+1)} \end{aligned}$$

$$> S_k$$

$$\therefore S_{k+1} > \frac{13}{24}$$

Q42

$$a_1 = \frac{1}{2} \left(a_0 + \frac{A}{a_0} \right), a_2 = \frac{1}{2} \left(a_1 + \frac{A}{a_1} \right) \text{ and } a_{n+1} = \frac{1}{2} \left(a_n + \frac{A}{a_n} \right)$$

$$\text{Let } P(n) : \frac{a_n - \sqrt{A}}{a_n + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}} \right)^{2^{n-1}}$$

For $n = 1$

$$\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}} \right)^{2^{0}}$$

$$\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}} \right)$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$

$$\frac{a_k - \sqrt{A}}{a_k + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}} \right) \quad \text{---(i)}$$

We have to show that

$$\frac{a_{k+1} - \sqrt{A}}{a_{k+1} + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}} \right)^{2^k}$$

$$\left(\frac{a_{k+1} - \sqrt{A}}{a_{k+1} + \sqrt{A}} \right)^{2^0}$$

$$= \left[\frac{\frac{1}{2} \left(a_k + \frac{A}{a_k} \right) - \sqrt{A}}{\frac{1}{2} \left(a_k + \frac{A}{a_k} \right) + \sqrt{A}} \right]^{2^0}$$

$$= \left[\frac{(a_k)^2 + A - 2a_k \sqrt{A}}{(a_k)^2 + A + 2a_k \sqrt{A}} \right]^{2^0}$$

$$= \frac{(a_k - \sqrt{A})^2}{(a_k + \sqrt{A})^2}$$

$$= \left[\frac{a_k - \sqrt{A}}{a_k + \sqrt{A}} \right]^{2^1}$$

$$= \left[\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}} \right]^{2^k}$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in \mathbb{N}$ by PMI

Q43

$$P(n) : 2^n \geq 3n$$

It is given that $P(r)$ is true, so

$$2^r \geq 3r \quad \text{--- (1)}$$

Multiplying both the sides by 2,

$$2^r \cdot 2 \geq 3r \cdot 2$$

$$2^{r+1} \geq 6r$$

$$2^{r+1} \geq 3r + 3r$$

$$2^{r+1} \geq 3 + 3r \quad \text{[Since } 3r \geq 3, 6r \geq 3 + 3r\text{]}$$

$$2^{r+1} \geq 3(r+1)$$

So, $P(r+1)$ is true

But for $r = 1$

$$2 \geq 3$$

It is true, so

$P(n)$ is not true for all $n \in N$ by *PMI*

Q44

$$S_n = 1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + \dots$$

Using induction we first show this is true for $n=2$,

$$\text{we get } S_2 = 1^2 + 2 \times 2^2 = 1 + 8 = 9$$

$$\text{From RHS, we have if } n \text{ is even } S_n = \frac{n(n+1)^2}{2}$$

$$S_2 = \frac{2 \times 9}{2} = 9$$

Now using induction we first show this is true also

$$\text{for } n=3, \text{ we get } S_3 = 1 + 8 + 9 = 18$$

$$\text{From RHS, we have if } n \text{ is odd } S_n = \frac{n^2(n+1)}{2}$$

$$S_3 = \frac{9 \times 4}{2} = 18$$

Lets assume above is true for $n=k$, we get

$$k \text{ is even, } S_k = 1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + \dots + 2 \times k^2 \text{ ---1}$$

$$k \text{ is odd, } S_k = 1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + \dots + k^2 \text{ ----2}$$

Now lets prove for $n=k+1$

If k is even, $k+1$ is odd we get

$$S_{k+1} = 1^2 + 2 \times 2^2 + 3^2 + \dots + 2 \times k^2 + (k+1)^2 \text{ ---3}$$

From above relation, we get

$$S_k = 1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + \dots + 2 \times k^2 = \frac{k(k+1)^2}{2}$$

Substitute this in 3, we get

$$S_{k+1} = \frac{k(k+1)^2}{2} + (k+1)^2 = \frac{(k+1)^2(k+2)}{2}$$

= RHS (when ' $k+1$ ' is odd)

Hence Proved

Q45

Let $P(n)$ be the statement given by

$P(n)$: The number of subsets of a set containing n distinct elements is 2^n
for all $n \in \mathbb{N}$.

Step I:

$$P(1): 2^1 = 2$$

For any set A containing 1 element, empty set and set A are two sets always subsets of A .

$\therefore P(1)$ is true.

Step II:

Let $P(m)$ is true. Then,

A set containing m distinct elements has 2^m subsets.....(i)

We have to prove that $P(m+1)$ is true.

Let the set A has $(m+1)$ elements.

$$A = \{1, 2, \dots, m, m+1\}$$

$$A = \{1, 2, \dots, m\} \cup \{m+1\}$$

Now using (i) we can say that $\{1, 2, \dots, m\}$ being m elements has 2^m subsets.

For $\{m+1\}$, empty set and set itself $\{m+1\}$ are subsets.

So, $\{m+1\}$ has 2 subsets.

$\Rightarrow$ Set A has $2^m + 2$ subsets

$\Rightarrow$ Set A has 2^{m+1} subsets

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in \mathbb{N}$.

Q46

Let $P(n)$ be the statement given by

$$P(n) : a_n = 3 \times 7^{n-1} \text{ for all } n \in \mathbb{N}.$$

Step I:

$$P(2) : a_2 = 3 \times 7^{2-1} = 21$$

Given that $a_k = 7 a_{k-1}$ for all natural numbers $k \geq 2$

$$a_2 = 7a_1 = 7 \times 3 = 21$$

$\therefore P(2)$ is true.

Step II:

Let $P(m)$ is true. Then,

$$a_m = 3 \times 7^{m-1} \dots\dots\dots (i)$$

We have to prove that $P(m+1)$ is true.

$$a_{m+1} = 7 a_m$$

$$a_{m+1} = 7 \times a_m$$

$$a_{m+1} = 7^1 \times 3 \times 7^{m-1} \dots\dots\dots [\text{from (i)}]$$

$$a_{m+1} = 3 \times 7^{m-1+1}$$

$$a_{m+1} = 3 \times 7^m$$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in \mathbb{N}$.

Q47

Let $P(n)$ be the statement given by

$$P(n) : x_n = \frac{2}{n!} \text{ for all } n \in \mathbb{N}.$$

Step I:

$$P(2) : x_2 = \frac{2}{2!} = 1$$

Given that $x_k = \frac{x_{k-1}}{n}$ for all natural numbers $k \geq 2$

$$x_2 = \frac{x_1}{2} = \frac{2}{2} = 1$$

$\therefore P(2)$ is true.

Step II:

Let $P(m)$ is true. Then,

$$x_m = \frac{2}{m!}, \dots \dots \dots (i)$$

We have to prove that $P(m+1)$ is true.

$$x_{m+1} = \frac{x_{m+1-1}}{m+1}$$

$$x_{m+1} = \frac{x_m}{m+1}$$

$$x_{m+1} = \frac{\frac{2}{m!}}{m+1} \dots \dots \dots [\text{from (i)}]$$

$$x_{m+1} = \frac{2}{m!(m+1)}$$

$$x_{m+1} = \frac{2}{(m+1)!}$$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in \mathbb{N}$.

Q48

Let $P(n)$ be the statement given by

$$P(n) : x_n = 5 + 4n \text{ for all } n \in \mathbb{N}.$$

Step I:

$$P(1) : x_1 = 5 + 4(1) = 5 + 4 = 9$$

Given that $x_k = 4 + x_{k-1}$ for all natural numbers k

$$x_1 = 4 + x_0 = 4 + 5 = 9$$

$\therefore P(1)$ is true.

Step II:

Let $P(m)$ is true. Then,

$$x_m = 5 + 4m \dots\dots\dots (i)$$

We have to prove that $P(m+1)$ is true.

$$x_{m+1} = 4 + x_{m+1-1}$$

$$x_{m+1} = 4 + x_m$$

$$x_{m+1} = 4 + 5 + 4m \dots\dots\dots [\text{from (i)}]$$

$$x_{m+1} = 5 + 4(m+1)$$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in \mathbb{N}$.

Q49

Let $P(n)$ be the statement given by

$$P(n) : \sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} \text{ for all natural numbers } n \geq 2.$$

Step I:

$$P(2) : \sqrt{2} = 1.4142$$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{1}{1.4142} = 1 + 0.7071 = 1.7071$$

$$\therefore \sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$$

$\therefore P(2)$ is true.

Step II:

Let $P(m)$ is true. Then,

$$\sqrt{m} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{m}} \dots \dots \dots (i)$$

We have to prove that $P(m+1)$ is true.

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{m}} > \sqrt{m} \dots \dots \dots [\text{from (i)}]$$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} > \sqrt{m} + \frac{1}{\sqrt{m+1}}$$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} > \frac{\sqrt{m^2 + m + 1}}{\sqrt{m+1}}$$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} > \frac{\sqrt{m^2 + 1}}{\sqrt{m+1}}$$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} > \frac{m+1}{\sqrt{m+1}}$$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} > \sqrt{m+1}$$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in \mathbb{N}$.

Q50

The distributive law from algebra states that for all real numbers c , a_1 and a_2 , we have $c(a_1 + a_2) = ca_1 + ca_2$. Use this law and mathematical induction to prove that, for all natural numbers, $n \geq 2$, if $c(a_1 + a_2 + \dots + a_n)$

$$= ca_1 + ca_2 + \dots + ca_n$$

Let $P(n)$ be the statement given by

$P(n) : c(a_1 + a_2 + \dots + a_n) = ca_1 + ca_2 + ca_3 + \dots + ca_n$ for all natural numbers $n \geq 2$.

Step I:

$$P(2) : c(a_1 + a_2) = ca_1 + ca_2$$

$\therefore P(2)$ is true.

Step II:

Let $P(m)$ is true. Then,

$$c(a_1 + a_2 + \dots + a_m) = ca_1 + ca_2 + ca_3 + \dots + ca_m \dots \dots \dots (i)$$

We have to prove that $P(m+1)$ is true.

$$c(a_1 + a_2 + \dots + a_m + a_{m+1}) = c[(a_1 + a_2 + \dots + a_m) + a_{m+1}]$$

$$c(a_1 + a_2 + \dots + a_m + a_{m+1}) = c(a_1 + a_2 + \dots + a_m) + ca_{m+1}$$

$$c(a_1 + a_2 + \dots + a_m + a_{m+1}) = ca_1 + ca_2 + ca_3 + \dots + ca_m + ca_{m+1} \\ \dots \dots \dots [\text{from (i)}]$$

$\Rightarrow P(m+1)$ is true.

Hence by the principle of mathematical induction, the given result is true for all $n \in \mathbb{N}$.